QA
457
S6

UC-NRLF

5B 278 823

GIFT OF
The Argonaut.

EX LIBRIS

759

Digitized by the Internet Archive
in 2007 with funding from
Microsoft Corporation

SOLID GEOMETRY

DEVELOPED BY THE

SYLLABUS METHOD

BY

EUGENE RANDOLPH SMITH, A.M.

HEADMASTER, THE PARK SCHOOL, BALTIMORE, MARYLAND

(FORMERLY HEAD OF THE DEPARTMENT OF MATHEMATICS, POLYTECHNIC
PREPARATORY SCHOOL, BROOKLYN, N.Y.)

IN CONSULTATION WITH

WILLIAM H. METZLER, Ph.D.

DEAN OF THE GRADUATE SCHOOL, AND HEAD OF THE DEPARTMENT
OF MATHEMATICS, SYRACUSE UNIVERSITY

NEW YORK .. CINCINNATI .. CHICAGO
AMERICAN BOOK COMPANY

Q 4 45 7
56

COPYRIGHT, 1913, BY
EUGENE RANDOLPH SMITH.
COPYRIGHT, 1913, IN GREAT BRITAIN.

SMITH SYL. SOLID GEOM.

W. P. I

Copyright

Smith Syl. Solid Geom.

PREFACE

THE author has always believed that the teaching of mathematics should encourage original thinking on the part of the pupils, and that a maximum of thought is difficult to obtain when the pupil is furnished with a text containing the proofs of the geometrical theorems in synthetic form. He has been using the heuristic method for thirteen years, and has proved to his own satisfaction that the average pupil can enter intelligently into the class development of new propositions, and, after a certain amount of training, can analyze propositions of ordinary difficulty with little or no help from the teacher.

This book has been written to encourage what is often called the "genetic," "heuristic," or "syllabus" method of teaching, but what is, in effect, simply the development of new work, partly in class, the rest by assignment as original exercises for preparation before the recitation. It is intended to contain all that a pupil really needs, except that which the teacher, and only the teacher, can best supply. While the proofs are not in the book in full and formal fashion, the author has given as much analysis, suggestion, and guidance as he feels to be wise. In fact, he believes that even this amount of help should often not be used until after the class discussion has covered the topic.

The list of propositions is based on the recommendations of the various committees, and especially on that of the National Geometry Committee of Fifteen. An attempt

has been made to reduce the list to a *pedagogical minimum*, without injuring the subject by over-reduction. The space thereby saved has been devoted to valuable matter not often found in geometries, such as the preliminary chapter, the detailed summaries, the Appendix, and the college examination questions.

Solid geometry is an especially promising field for heuristic methods because the pupils studying it are of sufficient maturity to enable them to think with some assurance and originality. If a pupil is ever to obtain the proper training in logical methods of thinking, it ought not to be longer delayed. Besides this, the results in solid geometry in schools and colleges show that former methods have not proved entirely satisfactory. This condition is probably due to the fact that the student is often plunged into the subject with little or no understanding of the figures with which he is to deal, the method he should use in studying them, or the relation of this new subject to the mathematics that has gone before. This book attempts to better this condition by introducing a preliminary chapter that *shows the relation between plane geometry and solid geometry*, and by building up the solid geometry figures in easy gradation from points, lines, and planes. This arrangement will give the pupil a clear understanding of the formation of the figures with which he is dealing and will lead him, in progressive stages, from the less complicated figures and proofs to those of greater difficulty. Incidentally, many of the difficult parts of the subject have been simplified, and the pupil is given a broad working knowledge of the properties and formulas of solids.

There are three combinations of related parts of the subject into single sections: prisms and cylinders, pyra-

mids and cones, and polyhedral angles and spherical polygons. The simplification of the subject by these combinations is quite marked, and will be evident on inspection of the sections in question. In the last case, it gives opportunity for some consideration of the way in which each can be made to depend on the other.

Some use has been made of two of the most powerful propositions of solid geometry, Cavalieri's Theorem, and the Prismatoid Formula, but only as alternate methods. The National Geometry Committee recommends that these two theorems be used but not proved. The Prismatoid Formula is used for spherical segments.

The use of limits and the incommensurable case has been left to the discretion of the teacher. The best present-day thought seems to favor the omission of as much of this work as possible.

The exercises are of wide variety in subject and difficulty, and include a set of about two hundred and seventy-five college examination questions chosen from recent papers set by various colleges, the New York State Board of Regents, and the College Entrance Examination Board. The exercises are arranged, some after the theorems to which they apply, and the rest in general sets.

While the author has made use of all the knowledge of solid geometry that his study and teaching have given him, still, as this text was written chiefly from classroom experiment, it is impossible to give credit specifically to authors from whom the germs of ideas may have been received. He is greatly indebted to Dr. William H. Metzler of Syracuse University, for his painstaking reading of the manuscript and for his valuable criticisms. Mr. Howard F. Hart of the Montclair High School, who has been trying in his classes some of the ideas that appear in

this book, has given the author the benefit of his experience with them, as well as a number of suggestions on other parts of the book. Mr. Clarence P. Scoboria of the Polytechnic Preparatory School, Brooklyn, also has aided in the preparation of the book by using the manuscript in class, and so making a practical test of the ideas represented. Besides those mentioned, the author wishes to express his obligation to many others who have taken an active interest in the preparation of the book.

EUGENE RANDOLPH SMITH.

CONTENTS

PRELIMINARY CHAPTER

	PAGES
HOW TO STUDY SOLID GEOMETRY	193
I. THE USE OF PLANE GEOMETRY IN SOLID GEOMETRY	194-202
II. METHODS OF ATTACK	203-215
III. THE REPRESENTATION OF SOLID GEOMETRY FIGURES	216-231
NOTES	232

BOOK VI.—LINES AND PLANES

I. THE PLANE	233-236
II. RELATIVE POSITION OF LINES AND PLANES IN SPACE; PARALLELS AND INTERSECTIONS	237-248
III. LINES PERPENDICULAR TO PLANES	249-253
IV. ANGLES BETWEEN PLANES	254-262
V. LOCUS OF POINTS	263-267
SUMMARY OF PROPOSITIONS	267-272
ORAL AND REVIEW QUESTIONS	272-274
GENERAL EXERCISES	274-277

BOOK VII.—POLYHEDRONS, CYLINDERS, AND CONES

I. THE PRISM AND THE CYLINDER	278-292
II. THE PYRAMID AND THE CONE	293-304
SUMMARY OF PROPOSITIONS	305-307
ORAL AND REVIEW QUESTIONS	308
GENERAL EXERCISES	308-313

BOOK VIII. — POLYHEDRAL ANGLES AND THE SPHERE

	PAGES
I. DEFINITIONS; SECANTS AND TANGENTS . . .	314-324
II. SPHERICAL POLYGONS AND POLYHEDRAL ANGLES .	325-343
III. AREA ON A SPHERICAL SURFACE . . .	344-349
IV. VOLUME OF A SPHERE AND ITS PARTS . . .	350-357
SUMMARY OF PROPOSITIONS . . .	357-361
ORAL AND REVIEW QUESTIONS . . .	362-364
GENERAL EXERCISES . . .	364-366

GENERAL

SUMMARY OF THE FORMULAS OF SOLID GEOMETRY .	367-369
COLLEGE EXAMINATION QUESTIONS . . .	370-392

APPENDIX

LOGIC	393
CAVALIERI'S THEOREM	394
THE PRISMATOID	396
CIRCULAR CYLINDRICAL AND CONICAL SURFACES . .	397
COMPARISON BETWEEN PLANE GEOMETRY AND SPHERICAL GEOMETRY	398
THE GENERAL POLYHEDRON	400
RADIAN MEASURE	401
SYMMETRY	403

INDEX

Light-faced numbers refer to pages and full-faced numbers to articles.

Acute angles	70	Cones	151, 158
Addition of lunes	286	spherical	303
Adjacent angles	70	Congruent polyhedral angles	265
Algebraic analysis	209	Congruent spherical polygons	264
Altitude	111, 155, 280	Conical space	147
Analysis	203-209	Conical surfaces	145, 321
Analytical geometry	266	Construction	31
Angles,		Contact, point of	196, 197, 233
between arcs	221	Converse of statement	211, 318
dihedral	64	Convex figures	102, 238
face	100	Coplanar figures	11
line with plane	83	Corresponding angles and	
of spherical polygon	237	sides	258
plane or measuring	65	Corresponding sets of prisms	172
spherical	222	Counter-clockwise motion	260
Area	97, 110, 151	Cubes	128, 253
Attack, methods of	203	Cuboids	127
Axioms	6, 18, 282	Cylinders	110, 114
Axis	115, 157, 204	Cylindrical space	105
Bases	110, 151, 154	Cylindrical surfaces	104, 321
Cavalieri bodies	137	Decahedron	98
Cavalieri's Theorem	137, 319	Degree of formulas	311
Center lines	232	Descriptive geometry	78
Center sects	232	Determined surfaces	4, 9
Center of a sphere	187	Diagonals	129
Central polyhedral angles	241	Diameters	189
Circles, of spheres	203, 207	Dihedral angles	64
Circular cones	158	Dimensions	129
Circular cylinders	114	Directrix	104, 145
Circular cylindrical and		Distance	
conical surfaces	321	on a sphere	248
Circumscribed figures		point to a plane	80
polyhedrons	201	Dodecahedrons	98, 255
prisms	116	Duplication of the cube	332
pyramids	159	Edges	
sets of prisms	172	64, 97, 99, 110, 151, 154, 312	
spheres	195	Elements	104, 145
Classification	203	Equality of lunes	287
Clockwise motion	260	Equilateral triangles	239
Closed surfaces	105, 147	Equivalence	124
Coincidence	263	Excess, spherical	278
Complementary angles	70	Explementary angles	70

Face angles	100	Negative of statement	210
Faces 64, 97, 98, 99, 110, 151, 154		Non-coplanar figures	11
Figures, representation of	216	Normal lines and planes	52
Foot, of line	14	Notation	
Frustums	153	of lunes	288
Generation		of spherical polygons	299
of a conical surface	145	Oblique, to a plane	52
of a cylindrical surface	104	Oblique cylinders and prisms	110
of a prismatic surface	104	Oblique parallelepipeds	127
of a pyramidal surface	145	Oblique sections	106
of a sphere	188	Obtuse angles	70
of a spherical surface	279	Obverse of statement	210
of a zone	281	Octahedrons	98, 254
Generatrix	104, 145	Opposite polyhedral angles	257
Great circles	207	Opposite spherical polygons	258
Hexahedrons	98, 253	Parallel figures	
Icosahedrons	98, 256	lines and planes	14
Inscribed figures		sections	106, 148
polyhedrons	195	summary of parallels	41, 43
prisms	116	three planes	39
pyramids	159	two planes	21
sets of prisms	172	Parallelepipeds	125, 127
spheres	201	Parts	100
Intersecting planes	17	Pentahedrons	98
Intersection of loci	207	Perpendiculars	
Isosceles triangles	239	lines and planes	52
Lateral area	110, 151	two planes	70
Lateral faces and edges	110, 154	Plane angles	65
Lateral surfaces	110, 151	Plane geometry, relation to	
Limit		solid geometry	193, 322
of circumscribed poly-		Planes,	
hedron	300	construction of	31
of inscribed polyhedron	282	defined	5
of prism	116	intersecting	18, 34, 35, 38
of pyramid	159	lines in	6, 22
of sets of prisms	175	lines parallel to	41, 43
Loci	88	lines perpendicular to	50, 53
Logic	210, 318	parallel	21, 39, 41
Lunes	259, 285-296	relative positions of	15, 33
Material spheres	227-230	representation of	10
Measuring angles		Polar chords	215
of dihedral angles	65, 69	Polar distances	215
of spherical angles	225	Polar triangles	273
Midsections	169	Poles	204
Negative converse of state-		Polygons, spherical	237
ment	210	Polyhedral angles	
		99, 240, 241, 244, 257	
		Polyhedrons 97, 103, 195, 250, 323	
		Prismatic space	105
		Prismatic surface	104

Prismatoids, defined	181	Sphere, defined	187
formula	182	Spherical excess	278
proof	320	Spherical figures	
Prisms, defined	110, 113	angles	222, 225, 226
inscribed and circum-		cones	303
scribed	172	polygons	237, 258
Projections	77	pyramids	305
Pyramidal space	147	sectors	303
Pyramidal surface	145	segments	309, 310
Pyramids	151, 156, 157	surfaces	187
spherical	305	triangles	239
		wedges	307
Quadrant	217	Squared paper	217
Quadrantal triangles	342	Straight angles	70
		Subtraction of lunes	286
Radian measure	324	Summaries	95, 185, 315, 317
Radius		Supplementary angles	70
of a circular cone	158	Surfaces	3, 97, 104, 145, 187
of a circular cylinder	115	Symmetric figures	325
of a sphere	189	Symmetric polyhedral angles	260, 265
Ratio of solids	131	Symmetric spherical polygons	260, 264
Reflex angles	70	Synthesis	203
Relative positions			
of line and plane	14	Tangent	
of line and sphere	196	to a cone	162
of plane and sphere	197	to a cylinder	118
of point and sphere	193	to a sphere	196, 197
of three planes	33	two spheres	233
of two lines	13	Tetrahedrons	98, 252
of two planes	15	Transversals	28
Right angles	70, 239	Triangles, spherical	239
Right cones	158	Trirectangular triangles	348
Right cylinders	110, 114	Truncated figures	152
Right parallelepipeds	127		
Right prisms	110	Ungula	307
Right sections	106	Unit of volume	135
Scalene triangles	239	Vertical dihedral angles	70
Secants	196, 197	Vertical polyhedral angles	257
Sections	101, 106, 148	Vertices	97, 99, 146, 237
Sectors, spherical	303	Volume	97, 135
Segments, spherical	309		
Similar solids	312	Wedge	312
Slant height	157, 158	spherical	307
Solid angles	99		
Solid geometry	193, 8, 322	Zones	280, 281
Solids, defined	2		
Space	1		

SUMMARY OF GEOMETRICAL SIGNS

$+$ plus, sign of addition	$<$ is less than
$-$ minus, sign of subtraction	$\nless$ is not less than
$\times$ times, sign of multiplication	$\parallel$ is parallel to
$\div, /, :,$ divided by, sign of division	$\perp$ is perpendicular to
$\sqrt{\quad}$ square root sign	$\angle$ or $\sphericalangle$ angle
$=$ is (or are) equal, or equivalent	$\triangle$ triangle
$\neq$ is not equal, or equivalent, to	$\parallel\!\!\!\diagup$ parallelogram
$\equiv$ is identical to	$\square$ rectangle
$\cong$ is congruent to	$\square$ square
$\doteq$ approaches as a limit	$\odot$ circle
$\sim$ is similar to	$\frown$ arc
$>$ is greater than	$\therefore$ therefore
$\nless$ is not greater than	$\because$ because, since

The signs for figures become plural by the addition of *s*, often within the sign, as $\square s$ for rectangles.

SOLID GEOMETRY

PRELIMINARY CHAPTER

HOW TO STUDY SOLID GEOMETRY

The Relation between Solid Geometry and Plane Geometry. — Solid geometry is not an entirely new subject. It is simply an extension of plane geometry, with which it has much in common. It is true that the conditions are somewhat changed by the use of figures not entirely in one plane, but the large part of solid geometry that treats of single planes of a solid figure introduces very little that is really new, since propositions of plane geometry obviously hold for any one plane of a figure. Even in the part of the subject where planes are least used, the definitions, axioms, and propositions of plane geometry are constantly applied, either directly or indirectly. As the methods of attacking a new proposition in solid geometry are the same as those used in plane geometry, the solid geometry propositions are made to depend, as far as possible, directly on plane geometry propositions.

In order that the pupil may find the transition to figures of three dimensions as easy as possible, this preliminary chapter gives some directions about methods of work. It contains a discussion of the plane geometry propositions that can be used in solid geometry, a review of methods of attack, and a short description of a method for drawing solid geometry figures. The section on drawing figures can best be used as the subject is studied, the different figures being examined as the pupil needs them in his work.

SECTION I. THE USE OF PLANE GEOMETRY IN SOLID GEOMETRY

In starting solid geometry, all of plane geometry is at the disposal of the student, but great care must be taken that it is used only for those figures for which the plane geometry proofs will hold. For example, it has been proved in plane geometry that *Lines parallel to the same line are parallel to each other*; but as this was proved only for the case when the three lines were in one plane, it cannot be used for the general case in solid geometry. It cannot be used, for instance, to prove that the back edge of the ceiling of a room is parallel to the front edge of the floor because they are both parallel to the front edge of the ceiling.

The proofs of some propositions in plane geometry would hold even if the figures used did not lie entirely in one plane; as, for example, congruence proofs. In a second kind of proposition, as in theorems concerning triangles, the figures, from the conditions of the proposition, necessarily lie in one plane. Such propositions can be used wherever their figures occur in solid geometry. In a third kind of proposition, where there is nothing in the conditions to show whether or not the figure is in one plane, it must be proved to lie in one plane before the plane geometry proof can be assumed to hold. An example is the proposition *Lines perpendicular to the same line are parallel*. By looking at the edges meeting at the corner of a room it will be seen that this proposition does not hold for the general case.

The propositions of plane geometry that are of most value in solid geometry will be summarized under the three heads indicated in the last paragraph.

A. SOME PARTS OF PLANE GEOMETRY THAT CAN BE USED IN SOLID GEOMETRY WHETHER OR NOT THE ENTIRE FIGURE IS IN ONE PLANE

- I. (a) All the definitions and axioms.
- (b) All right angles are equal; all straight angles are equal.
- (c) Complements, supplements, or explements of equal angles are equal.

II. FIGURES CONGRUENT.

- (a) The definition: two figures are congruent if they can be made to coincide.
- (b) Two triangles are congruent if they have three parts of one, of which at least one is a side, equal to the corresponding parts of the other; unless those three parts are two sides and an acute angle opposite one of them.

III. SIMILAR FIGURES.

- (a) Two polygons are similar (1) if their corresponding sides are proportional and their corresponding angles are equal; (2) if they are similar to the same polygon.
- (b) Two triangles are similar if they have (1) two angles equal; (2) two sides of one proportional to two sides of the other, and the included angle equal; (3) their sides correspondingly proportional.
- (c) Areas of similar figures are proportional to the squares of any two corresponding sects.

IV. OTHER COMPARISONS.

- (a) If two triangles have two sides of one equal to two sides of the other, then the third side of one is correspondingly greater than, equal to, or less than, the third side of the other, ac-

ording as its opposite angle is greater than, equal to, or less than, the opposite angle of the other, and conversely.

- (b) Areas of triangles having an angle of one equal to an angle of the other are to each other as the products of the including sides.

B. PLANE GEOMETRY PROPOSITIONS THAT CAN BE USED IN SOLID GEOMETRY BECAUSE THE NATURE OF THE FIGURE MAKES IT LIE ENTIRELY IN ONE PLANE

See § 9. Figures that lie entirely in one plane are a triangle, a line and a point outside, two intersecting lines, two parallel lines and therefore a parallelogram, and a circle.

I. PROPOSITIONS ABOUT A TRIANGLE.

- (a) If one side of a triangle is greater than, equal to, or less than, a second side, the angle opposite the first side is respectively greater than, equal to, or less than, the angle opposite the second side, and conversely.
- (b) The sum of two sides of a triangle is greater than the third side; their difference is less than that side.
- (c) The perpendicular bisectors of the sides of a triangle meet in a point (called the *circum-center*) equidistant from the vertices.
- (d) The bisectors of the interior angles of a triangle meet in a point (called the *incenter*) equidistant from the sides.
- (e) The medians of a triangle meet in a trisection point of each. In an equilateral triangle, the medians are also the altitudes.
- (f) A line parallel to one side of a triangle cuts the other sides proportionally, and conversely; it is in the same ratio to the side to

which it is parallel as the sects it cuts off on the other sides from their common vertex are to those sides.

- (g) In a right triangle with the altitude drawn to the hypotenuse, (1) the triangles are similar; (2) the altitude is the mean proportional between the sects of the hypotenuse; (3) each leg is the mean proportional between the hypotenuse and its own projection upon the hypotenuse.

II. PROPOSITIONS ABOUT PARALLELS AND PARALLELOGRAMS.

- (a) Definition: lines in the same plane that never meet are parallel.
- (b) Through a point there can be but one parallel to a given line.
- (c) If two parallels are cut by a transversal, two sets, each of four equal angles, are formed, the angles of one set being supplemental to the angles of the other set, and conversely.
- (d) In a parallelogram (1) the opposite sides are equal, (2) the opposite angles are equal, (3) either diagonal divides it into congruent triangles, (4) the diagonals bisect each other.
- (e) A quadrilateral is a parallelogram if (1) the opposite sides are parallel, (2) two opposite sides are equal and parallel, (3) the diagonals bisect each other.
- (f) Parallelograms having equal bases, and lying between the same parallels, are equivalent.
- (g) The sect between the midpoints of the legs of a trapezoid is one half the sum of the bases.

III. PROPOSITIONS IN REGARD TO A LINE AND AN EXTERNAL POINT.

- (a) There can be but one perpendicular from an external point to a line.
- (b) From an external point to a line, (1) the perpendicular is the shortest line; (2) obliques that make equal angles with the perpendicular or with the given line, or that cut off equal distances from the foot of the perpendicular, are equal, and conversely; (3) obliques that make unequal angles with the perpendicular, or with the given line, or that cut off unequal distances from the foot of the perpendicular, are unequal, the one that makes the greater angle with the perpendicular, or that cuts off the greater distance, being longer, and conversely.

IV. PROPOSITIONS ABOUT CIRCLES.

- (a) Radii, or diameters, of a circle are equal.
- (b) The diameter is the greatest chord; it bisects the circle and its circumference. If it is perpendicular to a chord, it bisects the chord, and conversely.
- (c) One, and but one, circle can be inscribed in, or circumscribed about a given triangle.
- (d) A central angle is measured by its arc; an inscribed angle, by one half its arc.
- (e) Equal chords subtend equal arcs, and are equidistant from the center, and conversely. The greater chord subtends the greater arc, and is nearer the center, and conversely.
- (f) A tangent is perpendicular to the radius drawn to the point of contact.

- (g) A tangent from a point to a circle is the mean proportional between the sects (from the point to the circumference) of any secant from the same point to the circle.
- (h) A circle can be inscribed in, or circumscribed about, any regular polygon.
- (i) If the number of sides of an inscribed or a circumscribed regular polygon is increased indefinitely, the perimeter and the area of the polygon approach the circumference and area of the circle as limits.

V. PROPOSITIONS ABOUT FIGURES FORMED BY TWO INTERSECTING LINES.

- (a) Any two vertical angles are equal.
- (b) Bisectors of vertical angles are in a straight line.
- (c) A quadrilateral is a parallelogram if its diagonals bisect each other.

VI. CONSTRUCTIONS. The given figure is in one plane, and the construction is to be done in that plane, except where it can be equally well done in a different plane, as in (d) and (e) below. Where the given figure can be in more than one plane, as in (a) and part of (c), the construction must be done in one of those planes. In every case there must be one plane in which the construction is done.

To construct

- (a) The perpendicular bisector of a sect.
- (b) The bisector of an angle.
- (c) The perpendicular to a line from or at a point.
- (d) A triangle, given its sides.
- (e) An angle equal to a given angle.
- (f) A line through a given point $\parallel$ to a given line.
- (g) The center of a circle, given an arc.

- (h) A circle, given an inscribed angle, and the chord it subtends; especially, the locus of the vertex of a right angle opposite a given hypotenuse.
- (i) A triangle equivalent to a given polygon.
- (j) A square equivalent to (1) the sum of two or more squares, (2) the difference of two squares, (3) any number of times a given square.
- (k) A third or a fourth proportional to given sects, or a mean proportional between them.
- (l) A regular polygon inscribed in, or circumscribed about, a given circle.

VII. FORMULAS.

- (a) *For the angles of triangles or other polygons.*
 - (1) An exterior angle of a triangle equals the sum of the non-adjacent interior angles. The sum of the angles of a triangle is a straight angle.
 - (2) The sum of the interior angles of a polygon of n sides is equal to $(n - 2)$ straight angles; the sum of its exterior angles is equal to two straight angles. Each angle of a regular polygon of n sides equals $\frac{(n - 2)}{n}$ straight angles.
- (b) *Concerning the sides of a right triangle.* The square of the hypotenuse of a right triangle equals the sum of the squares on the legs; the square of either leg equals the difference of the squares on the hypotenuse and the other leg.
- (c) The altitude of an equilateral triangle of side

a , is $\frac{a}{2}\sqrt{3}$; the altitude on b of a triangle of sides a, b, c , is

$$\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{a+b+c}{2}.$$

(d) The circumference of a circle is $2\pi r$, where r is the radius, and $\pi = 3.1416$ -.

(e) *Area.* The base is represented by b , by b_1 and b_2 when there are two bases; the altitude, by h ; the perimeter, by p ; the apothem, by a .

(1) Of a rectangle or other parallelogram, is bh .

(2) Of a triangle, is $\frac{bh}{2}$; of an equilateral

triangle of side a , is $\left(\frac{a}{2}\right)^2\sqrt{3}$; of a

triangle, using the notation in part (c),

is $\sqrt{s(s-a)(s-b)(s-c)}$.

(3) Of a trapezoid is $\frac{h}{2}(b_1 + b_2)$.

(4) Of a regular polygon is $\frac{1}{2}p \cdot a$.

(5) Of a circle is πr^2 , where r is its radius.

C. PLANE GEOMETRY PROPOSITIONS THAT CAN BE USED IN SOLID GEOMETRY ONLY WHEN THE ENTIRE FIGURE CAN BE SHOWN TO LIE IN ONE PLANE

I. CONCERNING PARALLELS AND PERPENDICULARS.

(a) There can be but one perpendicular to a given line at a given point. Note that in solid geometry any number of planes can contain the given line, and there can be one perpendicular to the line at the given point *in each of these planes*.

(b) A line perpendicular to one of two parallels is perpendicular to the other also.

(c) Lines perpendicular to the same line are parallel.

- (*d*) Lines perpendicular to intersecting lines must meet each other.

II. CONCERNING CIRCLES.

- (*a*) A line perpendicular to a radius at its end on the circumference is tangent to the circle.
- (*b*) Two circles are tangent if they meet at a point on their center line, or are tangent to the same line at the same point; they intersect twice if they meet at a point not on their center line.

III. LOCI. The given figure is in a plane. If, as in (*a*), (*e*), (*f*), and (*g*), it can be in more than one plane, one of those planes must be used. The locus of points *in that plane* in each case is :

- (*a*) Equidistant from two given points, the perpendicular bisector of the sect joining them.
- (*b*) Equidistant from three non-collinear points, the circumcenter of their triangle.
- (*c*) Equidistant from two intersecting lines, the bisectors of the angles between them; from two parallel lines, the third parallel midway between them.
- (*d*) Equidistant from three lines meeting in pairs, the incenter and three excenters of their triangle.
- (*e*) At a fixed distance from a given point, the circumference of a circle drawn with the point as center, and the distance as radius.
- (*f*) At a fixed distance from a given line, two parallels at that distance from the line.
- (*g*) Of the vertex of a given angle subtended by a given fixed sect, the two arcs of circles having that sect as a chord, and the angle inscribed on the arc of that chord; in particular, for a right angle, a circle on the sect as a diameter.

SECTION II. METHODS OF ATTACK

Synthesis and Analysis. The most usual form for a proof is *synthetic*, that is, the proof starts from the given conditions, and, step by step, builds on them until the conclusion is reached. The demonstrations of plane geometry, to which the pupil is accustomed, are of this form.

While this is an excellent way in which to state a proof that is already known, it is not adapted to discovering a proof, unless the conclusion is an almost immediate consequence of the conditions.

Usually a proof must be discovered by *analysis*, that is, by working back from the required conclusion to the conditions upon which the conclusion must depend. The different methods of analyzing a proposition will be taken up in some detail. The one which is most nearly pure analysis is called *the analysis method*, the others being named from some important feature of the process used.

Analysis by the Classification Method. The most generally useful of all the methods of attack is analysis by the *classification method*. It considers all propositions as belonging to *classes* according to their conclusions. For example, in using lines, one class proves sects equal, another proves them unequal, a third proves them proportional, while a fourth proves them parallel, etc.

It is evident then, that each new proposition must be proved by some already known proposition of its own

class,—the necessary definitions and axioms being included in their respective classes,—unless it can be proved by reasoning from its converse or its negative converse (see LOGIC, p. 210). As an illustration, the first proposition in solid geometry requiring that lines be proved parallel, must be proved by some plane geometry method of proving lines parallel. As a matter of fact, this proposition uses the definition of parallel lines. The second proposition requiring that lines be proved parallel must use a plane geometry method, *or that first proposition of solid geometry*, and so successive propositions build up a solid geometry class that proves lines parallel. The same idea holds throughout solid geometry, each proposition, unless pure logic is sufficient, being proved by a plane geometry proposition of the same class, or by a preceding solid geometry proposition of that class. In case a proposition is the first one of a new class, it uses the definitions and axioms upon which that class is founded.

When a proposition has been classified, and the preceding propositions of that class have been thought over, or written in a list, the given conditions should be carefully considered. The possible methods of proving the proposition that do not seem to fit the conditions should be eliminated, and a proof should be attempted by whichever of the remaining methods seems the best for the purpose. In many propositions each of the methods will give a proof. After the method of proof has been decided upon, the several steps are found by repeated applications of the classification method.

The use of this method of analysis is more easily seen by reference to the following illustrations, which are stated in condensed form. As the student has not yet

studied solid geometry, the illustrations are taken from plane geometry.

ILLUSTRATION 1. *The opposite sides of a parallelogram are equal.*

On considering the methods of proving sects equal, it appears that the most applicable method is by corresponding parts of congruent triangles; therefore a diagonal should be drawn to form the triangles.

On considering the methods of proving triangles congruent, it appears that it is impossible to use a theorem that requires two or three pairs of equal sides, since two pairs of the sides are to be proved equal. Therefore the congruence of the triangles must be proved by two angles and a side of one equal to the corresponding parts of the other.

The diagonal is the common side, and the use of the given condition (that is, the fact that the figure is a parallelogram, and the definition of a parallelogram) suggests the method of proving the angles equal. There are two ways to finish the proof, (1) by using two angles and the included side, (2) by using two angles and the side opposite one of them.

In this theorem there are three classifications, that is, at three points it must be decided how to accomplish the next step, and in each step the examination of what is wanted, and of what is known, shows what must be done.

This method of reasoning is not confined to geometry, but is used by the artisan, the engineer, and the business man who must meet certain conditions, and must decide on *what he needs to do, the various methods of doing it, and on the one method best applicable to the conditions under which he is working.*

ILLUSTRATION 2. *If a triangle is inscribed in a circle, the product of two of its sides equals the altitude to the third side times the diameter of the circle.*

Classification. Products can be proved equal by means of equivalent areas, or by a proportion. A proportion can be proved by similar triangles, since the various methods of proving a proportion can usually be reduced to the method by similar triangles. Since a circle is given, it can be used to measure equal angles in the triangles that are to be proved similar.

So far the classification has been merely mental consideration of the required conclusion and the given figure. Now that the discussion has reached the point of deciding to try to get similar triangles by the use of angles, the auxiliary (or construction) lines must be added to the figure. Let the triangle ABC , with the altitude CH to AB , be inscribed in circle O . To prove $BC \cdot CA = CH \cdot d$, or $\frac{BC}{d} = \frac{CH}{CA}$. Since CH and CA are in the triangle CAH , d must be placed so as to be in a triangle with BC , that is, it must be drawn from B or from C . In this figure C is chosen. To complete the triangle, draw the sect from B to the other end of the diameter X , then show that the

triangles CAH and BCX are similar, since they have two angles respectively equal.

Special Case of Analysis by the Classification Method : Intersection of Loci. Many propositions require that a locus be found that depends on two or three loci already known. This fact is discovered by the examination of the required conclusion; then the known loci are constructed. The figure formed by their common points, that is, the *intersection of the loci*, is the required locus.

For example: *To find the locus of points equidistant from two given points; and also equidistant from two given intersecting lines.*

The locus of points equidistant from two given points is the perpendicular bisector of the sect joining them, while the locus of points equidistant from two given intersecting lines is the two lines bisecting the angles between those lines. Therefore the required locus is the one or more points in which the perpendicular bisector meets the bisectors of the angles. This method is much used in solid geometry. Three loci are sometimes needed to determine the required locus.

To sum up the classification method of attack :

1. Consider the known methods of obtaining the required conclusion.
2. Examine those methods in relation to the given conditions, and eliminate the methods that seem least applicable.
3. Of the methods remaining choose the one that appears most applicable to reasoning from the condition to the conclusion. Classify as necessary during the proof; draw the auxiliary lines that the method used makes necessary.

WARNING. *Remember to use each fact of the given conditions in the proof. Since the conclusion follows only from the conditions, it is useless to attempt a proof that does not make use of all conditions. To use a condition it is necessary to make it the authority for at least one step of the proof.*

The Analysis Method. This method consists in considering the conclusion to be true, and reasoning from the conclusion to the conditions upon which it depends. Then the order of reasoning is reversed to prove the proposition. This method is of most use in constructions that cannot be readily done by the classification method.

ILLUSTRATION. *To draw a parallel to the base of an isosceles triangle such that its length between the legs of the triangle shall equal the length it cuts off (from the base) on either leg.*

Suppose that XY is parallel to AB , and is equal to XA and to BY . Because $XY = XA$, the drawing of AY forms an isosceles triangle, and $\angle XAY = \angle XYA$. Because XY is parallel to AB , $\angle XYA = \angle YAB$. That is, AY bisects $\angle A$. Hence the required line can be constructed by bisecting $\angle A$ by AY , and drawing a line from Y parallel to AB .

This and the classification method are evidently variations of one method. In this example, the classification method of proof would be: $XY = XA$ if $\angle XAY = \angle XYA$,

and $XY \parallel AB$ if $\angle XYA = \angle YAB$, both of which conditions are true if AY bisects $\angle A$. In such cases as this, analysis is probably simpler.

Algebraic Analysis. Many propositions and exercises can be solved by *algebraic analysis*. They are usually constructions or numerical propositions. The general method is to use letters for the unknown parts, and then to solve algebraically the equations that can be written in terms of these letters. If the question is a construction, the required element is constructed after it has been found in terms of the given parts of the figure.

ILLUSTRATIONS. (1) *If one leg of a right triangle is double another leg, and if the area is 9 square feet, find the lengths of the sides.*

Let the shorter leg be x , the longer $2x$, then $\frac{1}{2}x \cdot 2x = 9$, and the legs are 3 ft. and 6 ft.

(2) *To construct a circle whose area is three times that of a given circle.*

Call the radius of the given circle r , that of the required circle R . Then $\pi R^2 = 3\pi r^2$, and $R = r\sqrt{3}$. Construct R as the mean proportional between r and $3r$.

(3) *Find the radius of a circle such that the number of linear feet in the circumference equals the number of square feet in the area.*

If the radius is r , $2\pi r = \pi r^2$, and $r = 2$ feet.

(4) *Find by algebraic analysis how to cut a given sect in mean and extreme ratio.*

Let the given sect be a , and the required mean sect be x ; then, by definition of mean and extreme ratio, $\frac{a}{x} = \frac{x}{a-x}$, or $x^2 + ax - a^2 = 0$. On solving for x , $x = \frac{-a \pm a\sqrt{5}}{2}$.

Using the positive square root for internal division, (1) construct $a\sqrt{5}$ as the mean proportional between a and $5a$; (2) subtract a from $a\sqrt{5}$; (3) bisect the remainder; (4) cut the resulting sect off on a , and a will be cut in mean and extreme ratio.

The Use of Pure Logic.¹ For convenience, let the condition, or hypothesis, of a statement be represented by H , and the conclusion by C .

There are four closely related conditional statements, as follows:

- (1) If H is true, then C is true;
- (2) If H is not true, then C is not true;
- (3) If C is true, then H is true;
- (4) If C is not true, then H is not true.

Of these, (1) and (2) are *negatives*, or *obverses*, of each other, or either is said to be the *negative* of the other.

(1) and (3) are *converses* of each other, or either is said to be the *converse* of the other.

(1) and (4) are *negative converses* of each other, or either is said to be the *negative converse* of the other.

Evidently (3) and (4) are also negatives of each other, (2) and (3) are negative converses, etc.

LAWS CONCERNING THE TRUTH OF THE RELATED STATEMENTS, IF ONE OR MORE STATEMENTS ARE KNOWN TO BE TRUE

1. Reasoning from a Single True Statement.

If a statement is true, its negative converse is also true; its negative and its converse may, or may not, be true.

The truth of this law can be seen by the following reasoning: Suppose (1) is given, that is, If H is true, then C is

¹ See also Appendix, § 318.

true. Then if C is not true, either (a) H is true, in which case C is true, or (b) H is not true. But (a) contradicts the condition, so is impossible, and (b) must follow, that is, the negative converse of a true statement is also true.

But if C is true, H may be either true or untrue without bringing in a contradiction; and if H is untrue, C may be true or untrue without bringing in a contradiction, so the truth or untruth of a converse or a negative cannot be established from a single statement. If the converse or the negative is true, it is because of some one or more other propositions used in connection with the one to which it is related. If either the converse or the negative is true, both are true, for they are negative converses of each other. See also Law 2.

ILLUSTRATION. Suppose it is known that

(1) *If two lines crossed by a transversal meet, any pair of alternate angles are unequal.*

Then its negative converse must also be true, that is,

(2) *If a pair of alternate angles formed by two lines crossed by a transversal are equal, the lines are parallel.*

Without the use of one or more other geometrical statements, it cannot be known whether (3), its converse, or (4), its negative, is true or not.

(3) *If a pair of alternate angles formed by two lines crossed by a transversal are unequal, the lines meet (are not parallel).*

(4) *If two lines crossed by a transversal are parallel, any pair of alternate angles are equal.*

To Prove the Converse of a True Statement. Show that the figure constructed by using the condition and the figure constructed by using the conclusion *are identical*. For example, to prove (4) from (2);

If a figure is drawn with the angles equal, the lines will be parallel by (2); if the lines are drawn parallel, the same figure will result, because *Through a point there can be but one line parallel to a second line*, which is the other geometrical statement needed in this proof.

To Prove the Negative of a True Statement. Show that the figure constructed by using the condition and the figure constructed by using the conclusion *cannot be the same*. For example, to prove (3) from (2):

If a figure is drawn with the angles equal, the lines will be parallel by (2); if another line is drawn through the same point making the angles unequal, it will be a different line, and so cannot be parallel since there can be but one parallel through the point. To prove the conclusion, it is necessary to use the additional geometrical principle that but one parallel to a given line can be drawn through a given point.

WARNING. *It must not be assumed that the converse and the obverse of a proposition are even likely to be true; if they are true, they can be proved as in the examples just given, or by the method of reasoning from statements covering all possibilities, which will be given later.*

NOTE. It should be observed in these examples of the methods used in proving a converse or a negative, that a line is added to the figure, and is then proved to coincide or not to coincide with one of the given lines. Nothing is assumed in constructing it as to whether or not it is the same line as the one in the figure, but if it is proved to be the same line, the two are then identical, and all properties of the one are also properties of the other.

The use of two lines which may be identical lines should not be confused with the use of two *given lines*, for all given elements are supposed to be distinct; as, when two lines are given, they are assumed to be different lines. Were it not for this understanding, the proposition, *Lines parallel to the same line are parallel to each other*, would need to be worded, *Two different lines that are parallel to the same line are parallel to each other*.

The method of reasoning from the negative converse is very common throughout mathematics, as well as wherever reasoning is used in everyday life. The most common method of proving a proposition or other fact is to show :

(1) That a certain condition must give a certain conclusion (which it may require one or more steps of reasoning to establish) ;

(2) That the conclusion is not true.

From this it follows by the negative converse law that the condition could not have been true.

A few examples outside of mathematics follow :

1. If it were freezing, the puddles would be covered with ice. There is no ice on this puddle, so it is not freezing.

2. If the office boy had been attending to his work, the waste paper basket would have been emptied. It is not emptied, so he has not been attending to his work.

3. If this window had been broken from the inside, the glass would have fallen out. It has fallen inside, so the glass was broken from the outside.

Other examples could be given without limit. In mathematics, when several steps are used to reason from the condition to an untrue conclusion, and so to show that the condition must have been untrue, the method of reasoning is sometimes called *reductio ad absurdum*.

2. Reasoning from True Statements Covering all Possibilities.

If statements such that their conditions cover all possibilities are true, and no two of their conclusions can be true at once, then their converses are also true.

For example, suppose it has been proved that

(1) *If two sides of a triangle are equal, the opposite angles are equal* ; and

(2) *If two sides of a triangle are unequal, the opposite angles are unequal, the larger angle being opposite the longer side.*

Then, since the conditions with regard to the sides cover all possibilities, and only one of the conclusions can be true, the converses of these statements must follow. Stated with regard to triangle ABC ,

Given; If $BC \geq CA$, then correspondingly $\angle A \geq \angle B$. It follows that if $\angle A \leq \angle B$, then correspondingly $BC \leq CA$.

If the explosion of the battleship Maine was within the ship, the plates must have been blown outward; if it was outside the ship, the plates must have been blown inward; then the converses of these statements are also true, and since it was found that the plates were bent in, the explosion must have been outside the ship.

The truth of converse statements may also be shown by *elimination of possibilities*, that is by showing that all but one of the possibilities are untrue, and that the one remaining is therefore the true one because it is *the only remaining possibility*. For instance, in the foregoing geometrical illustration, given the same conditions, to find what is true if $\angle A > \angle B$.

If BC were equal to CA , then $\angle A$ would be equal to $\angle B$, which is not true.

If BC were $< CA$, then $\angle A$ would be $< \angle B$, which is not true; therefore BC is $> CA$, because that is the only remaining possibility.

Order and Arrangement of a Proof. In writing out a formal proof, it is best to have a definite order of arrangement that will show all the important details of the proof. The following arrangement is recommended for all propositions:

STATEMENT OF PROPOSITION

Diagram

Given. The conditions of the proposition.

To prove. The conclusion. (Both condition and conclusion should be in terms of the figure. If it is a construction, **Required** should take the place of **To prove**.)

Proof. 1. The proof should follow in numbered steps. (If it is a construction, the proof may be divided into **Construction** and **Proof**.)

ILLUSTRATION

A line and a point outside determine a plane.

Given. Line AB and external point P .

To prove. AB and P determine a plane.

- Proof.** 1. Let K and L be two points on AB , then K , L , and P determine a plane (definition of plane, § 5, p. 233).
2. But AB is entirely in this plane, for two of its points are in the plane (axiom of line and plane, § 6, p. 234).
3. Therefore AB and P determine a plane, for they lie in one, and but one, plane.

SECTION III. THE REPRESENTATION OF SOLID GEOMETRY FIGURES

Since in solid geometry three-dimensional figures are usually represented on the paper or on the blackboard, that is, on a surface of but two dimensions, the student needs some idea of how to make the figures appear solid. While there is neither space nor time for a study of drawing, it is believed that the following brief directions may be of assistance in giving this idea.

Unless the student has given some time to the study of perspective, the parallel line method will be the simplest for him to use. In this, a plane, or rather, — since a plane is unlimited in extent, — that portion of it shown in the figure, is, in general, represented by a parallelogram. It follows that many solids, since they are bounded by planes, can be drawn by the use of sets of parallel lines. This, of course, is not true of solids the conditions of which require that some portion of a plane shown in the figure shall be a circle, or a polygon other than a parallelogram. Where parallel lines are used, one set is usually drawn horizontally, the other somewhat inclined to the vertical.

There are a few general rules that are of assistance in drawing all the figures, the most important being the following :

The figures are usually supposed to be viewed from a point a little above, and either directly in front, or a little to one side.

Lines nearer to the observer should be somewhat heavier than those at the back of the figure.

As planes are supposed to be opaque, all lines apparently covered by them should be left out entirely, or, if needed in the figure, should be dotted. Where it seems desirable, lines that should not appear in the figure may be drawn and afterwards erased.

A circle, unless viewed from a point directly opposite the center, appears elliptical, the ellipse being narrower as the eye is nearer its plane.

It should be noted that two lines that intersect in a drawing do not necessarily intersect in the three-dimensional figure, and that the actual length of a line, or the size of an angle, may be very different from its representation, as each part is drawn so as to *appear* to be of the right size in the three-dimensional figure represented.

The Use of Squared Paper. At the beginning, squared paper will be found very convenient for drawing the solid geometry figures, for with it equal lines, parallel lines, and perpendicular lines can be drawn with no instrument but the straightedge. As the student gains skill in drawing he should not confine himself to this method, but should draw accurate figures on unruled paper, using compasses and ruler. He should also draw reasonably accurate freehand figures. If models of the solid figures are used in addition to the drawings, the two together will give the best possible idea of the three-dimensional figures.

The following explanation shows how to draw parallel or perpendicular lines on squared paper:

(1) **Parallel Lines.** (Fig. 1). Draw the lines so that they connect points the same distance apart, measured

along the lines of the paper, to the right or left, and up or down. For example, in Fig. 1, to draw a line from C parallel to AB , note that AB extends to the right 5 spaces, and up 15 spaces; therefore count from C to the right 5 spaces, and up 15 spaces to D , and draw CD . This also makes the lines equal, as can be seen by this illustration, where $CD = AB$. If the lines are not to be of the same length, the same method is used, and that part of CD that is needed is taken.

(2) **Perpendicular Lines.** (Fig. 2). Count as before, but draw the new line so that one direction is reversed, — that is, up for down, left for right, etc., — and so that the two numbers are interchanged. In Fig. 2, to draw a line from P perpendicular to AB , note that AB extends 10 spaces down, and 15 spaces to the right, so draw from P to a point O , 15 spaces down, and 10 spaces to the *left*, or, 15 spaces *up*, and 10 spaces to the right.

The Plane (Figs. 3, 4). Draw a parallelogram, usually with two edges horizontal. The other positions in which a plane most frequently appears will be taken up in the following figures. If, for any reason, it is desirable to emphasize the fact that the plane is unlimited in extent, one pair of edges can be broken, as in Fig. 4.

Two Lines in Space (Figs. 5, 8). Intersecting or parallel lines do not differ from those in plane geometry. Two lines not in the same plane, called non-coplanar lines, cannot be distinguished from intersecting lines, unless there is more in the figure to help show their relative position. In Fig. 5, the plane M appears to contain CD , and so makes it appear that AB and CD are not in the same plane. In Fig. 8, AB and RS are not in the same

PLATE I

FIG. 1. PARALLEL LINES

FIG. 2. PERPENDICULAR LINES

PLATE II

FIG. 3.

PLANES

FIG. 4

FIG. 5. TWO NON-COPLANAR LINES

FIG. 6. A LINE AND A PLANE

FIG. 7

plane, as is shown by the points in which they meet the planes M , N , and P .

A Line and a Plane (Figs. 5, 6, 7, 8, 9).

(1) **The Line in the Plane.** Draw it within the parallelogram (or other outline) that represents the plane, as AC and OB in Fig. 6.

(2) **The Line Parallel to the Plane.** Draw it parallel to a pair of sides of the parallelogram, preferably to the horizontal sides, as RS in Fig. 6. If more than one line is to be drawn through a point parallel to the plane, draw them so that they will be within the outline that represents another plane parallel to the given plane. Thus, in Fig. 8, OK and OL are parallel to plane P .

(3) **The Line Perpendicular to the Plane.** Draw it perpendicular to the pair of horizontal sides, as PO in Fig. 6, AB in Fig. 8, and PO in Fig. 9. If a portion of a plane is represented by an outline other than a parallelogram, imagine the parallelogram you would draw to represent that plane, and then draw the line so as to be perpendicular to two sides of that parallelogram. See $P'O'$ in Fig. 7.

(4) **The Line Oblique to the Plane.** Draw it so as not to be perpendicular to the horizontal sides, as PA , PB , PC in Fig. 6, BC and RS in Fig. 8, and PB in Fig. 9.

Parallel Planes (Fig. 8). Represent them by parallelograms with their corresponding sides parallel. To represent portions of parallel planes, if the conditions of the figure require some other outline, as in Fig. 22, the corresponding sides should still be parallel.

Intersecting Planes (Figs. 9, 10). First draw the line of intersection, preferably somewhat oblique to the vertical,

PLATE III

FIG. 8. PARALLEL PLANES

FIG. 9. INTERSECTING PLANES

so that the observer will appear to be looking at one plane from a little above, and at the other from a little to one side. Then fill in the outlines of the planes, keeping one pair of sides of the parallelogram representing each plane parallel to the intersection. If the planes are to be perpendicular to each other, make a pair of sides of one parallelogram perpendicular to a pair of sides of the other, as in Fig. 10; otherwise, let the sides not parallel to the intersection be oblique to each other, as in Fig. 9.

In Fig. 9, the plane PAO is perpendicular to the plane M and to the plane N , but, on account of its position with reference to those two planes, it cannot be shown in the same way as in Fig. 10. It is made to appear perpendicular by being drawn so as to appear parallel to the horizontal sides of the parallelogram representing plane M , and to the corresponding sides of the parallelogram representing plane N . Note that PA and AO are made to appear perpendicular to the intersection by being drawn parallel to a pair of edges. In the same way, in Fig. 10, PO and OA are perpendicular to the intersection, while OB is oblique to it.

Three Planes Intersecting in Parallel Lines (Figs. 11, 12). Draw the parallel intersections, dotting whatever is covered by the planes, as in Fig. 12. The planes can be represented by completing the parallelograms as in these figures, or by using a broken edge, as in Fig. 13.

For more than three planes intersecting in parallel lines, see Figs. 17, 18, 19.

Three or more Planes Concurrent in a Point (Figs. 13, 14). Choose the point that is to be the vertex, as V , and draw the edges, and fill in the other boundaries

PLATE IV

FIG. 10. PERPENDICULAR PLANES *FIG. 11* *FIG. 12**THREE PLANES INTERSECTING IN
PARALLEL LINES*

PLATE V

FIG. 13

FIG. 14

*THREE OR MORE PLANES INTERSECTING
IN A POINT*

FIG. 15

FIG. 16

*THREE OR MORE PLANES MEETING IN
A COMMON LINE*

afterwards, or draw the other boundary, and connect its vertices to the vertex V . Be careful that the edges behind the planes, as VE and VF in Fig. 14, do not lie so near the other edges as to cause confusion. The termination of the figure can be by a plane, as ABC in Fig. 13, or $ABCDEF$ in Fig. 14, or by a broken boundary, as in Fig. 13, when the fact that the figure is unlimited is to be emphasized.

Three or more Planes Meeting in a Common Line (Figs. 15, 16). Draw the intersection, then the plane nearest the observer, and so on; keep one side parallel to the intersection. The hidden lines can be dotted when they are needed in the figure.

Parallelepipeds and other Prisms (Figs. 17, 18, 19). Draw one base, then draw parallel equal edges, and fill in the other base last. A pentagonal prism is shown in Fig. 17, an oblique parallelepiped in Fig. 18, and a rectangular parallelepiped in Fig. 19.

The Cylinder (Fig. 20). Draw two parallel equal lines to represent two elements, then draw ellipses between their ends. $ABCD$ is a section of the cylinder through an element.

The Pyramid (Figs. 13, 14, 21, 22). The figures $V-ABC$ in Fig. 13, and $V-ABCDEF$ in Fig. 14 are pyramids and can be drawn as explained there. In Fig. 21 the pyramid is regular, as is indicated by its regular base, and by the fact that V is on the perpendicular at the center of the base.

PLATE VI

FIG. 17

FIG. 18

FIG. 19

PARALLELEPIPEDS AND OTHER PRISMS

FIG. 20
CYLINDER

FIG. 21

FIG. 22

PYRAMIDS

For a frustum, draw a pyramid, with the lateral edges filled in lightly so that they can be erased where they are not needed ; then from some point on one edge, as P in Fig. 22, draw a parallel section by drawing lines parallel to the corresponding edges of the base from each lateral edge to the next lateral edge. For example, PQ is parallel to BC , etc. The parts of the lateral edges from the upper base to the vertex should then be erased.

The Cone (Figs. 23, 24, 25). A cone can be drawn by drawing two lines from the point to be used as its vertex, and then filling in the elliptical base. Figure 23 is an oblique cone with a triangular section ABV ; Fig. 24 is a right cone with its altitude meeting the base at its center.

In drawing a frustum, draw two elements between parallel lines, that is, so that they would be legs of a trapezoid if their ends were connected, then fill in the elliptical bases, as in Fig. 25.

The Sphere (Figs. 26, 27, 28, 29). The sphere is represented by a circle, the appearance of solidity being given by the other lines drawn in the figure. Of these, circles on the surface of the sphere are the most frequent. In Fig. 26 a great circle G and a small circle S are shown, with their radii, and, since they are parallel, with their common axis PP' . All circles except the one used for the outline of the sphere should appear as ellipses, the ellipse being narrower as it is supposed to be looked at from a point more nearly in its plane, and more like a circle as it is supposed to be looked at from a point more nearly on its axis.

To draw the arc of a great circle, the arc should be so curved that it will appear to meet the outline circle at the ends of a diameter. In Fig. 27 the spherical triangle

PLATE VII

FIG. 23

FIG. 24

FIG. 25

CONES

FIG. 26

FIG. 27

THE SPHERE : SPHERICAL TRIANGLES

ABC is drawn, and the sides are continued in dotted lines, to show how their circles would appear to pass through the ends of diameters. In this figure, the observer is looking toward the center of the sphere from a point such that the line along which he is looking meets the surface inside the triangle, as is shown by the fact that the sides all curve away from the inside of the triangle. In Fig. 28 the eye is in such a position that the sides of the triangle all appear to curve away from a point outside of the triangle. It makes little difference in what position the eye is supposed to be, if the arcs are drawn through the ends of diameters so that they give the appearance of solidity to the sphere.

To draw polar triangles, draw the axis of each of the circles that form one triangle, using only that half of the axis that meets the same hemispherical surface on which the opposite vertex lies, and draw great circle arcs through the poles found, using the method shown in Figs. 27 and 28.

Other Figures. The diagrams shown are those most frequently used in solid geometry, the others of importance being for the most part combinations of these. For example, polyhedral angles are formed by three or more planes meeting in a point; a spherical pyramid is a combination of the pyramid with great circle arcs, the regular tetrahedron and the regular octahedron are respectively a pyramid, and two pyramids placed base to base, while a regular hexahedron is a rectangular parallelepiped with equal edges. If the pupil has clearly in mind the few principles by which these diagrams have been drawn, the building up of new and more complicated diagrams should not be found difficult.

PLATE VIII

FIG. 28. SPHERICAL TRIANGLES

FIG. 29. POLAR TRIANGLES

NOTES

Statements marked with an asterisk require proof. Such statements include many simple deductions from the definitions and axioms, not important enough to receive the emphasis given to propositions, as well as other important statements that follow so naturally from the discussion that a more or less informal proof of them seems most satisfactory. Both these classes of statements, which have been italicized, may be considered as corollaries of the definitions and axioms.

In order to avoid circumlocution and monotonous repetition in the statements concerning mensuration, this book, in speaking of the length of a line, usually omits "the length of," and in speaking of the areas of plane figures usually omits "the area of," wherever there is no doubt as to the meaning. For example, in the statement, "*The volume of a parallelepiped equals the product of its base and its altitude,*" "base" manifestly means "the area of the base" and, "altitude," "the length of the altitude," and should be so interpreted.

BOOK VI. LINES AND PLANES

SECTION I. THE PLANE

1. **Space.** The space in which everything exists is, as far as experience shows, unlimited. It is also divisible, for all the bodies with which we are familiar occupy portions of space. While the space studied in solid geometry (sometimes called Euclidean space) appears to be the same as that in which we exist, it is assumed only to be a space such that in it the axioms and postulates of geometry are true. Euclidean space is assumed to be unlimited in extent and to be divisible.

2. **Solids.** Any limited portion of space is called a *geometric solid*, or simply a *solid*. The term "solid" must not be confused with "solid body," for the geometric solid is studied as a part of space, and is entirely irrespective of any physical body that might occupy it.

3. **Surfaces.** That which separates one portion of space from an adjoining portion is called a *surface*. A surface may be limited or unlimited in extent, and can have limited portions.

4. **Determining a Surface.** As in plane geometry a straight line is said to be *determined* by any two of its points, so in solid geometry a surface of any specified kind is said to be *determined* by given points, lines, etc., if it is the only surface of its kind that contains them.

5. **Planes.** The surface that is determined by any three of its points that are not in a straight line is called

a *plane surface*, or simply a *plane*. A plane is unlimited in extent, but limited portions of a plane can be considered. Familiar examples of a portion of a plane are, the surface of a desk, the surface of a blackboard, or any surface commonly spoken of as “flat.”

6. Axiom of the Straight Line and the Plane. *If two points of a straight line are in a plane, the whole line is in the plane.*

7. Similarity between the Plane and the Straight Line. The plane occupies the same position among surfaces that the straight line does among lines; as indicated by the axiom, it is straight, —that is, a straight line could lie in it, —through any two of its points. Another similarity arises from the fact that the straight line is the one line determined by two of its points, while the plane is the one surface determined by three of its points.

8. Solid Geometry. Solid geometry treats of figures that do not lie entirely in one plane.

In proving the propositions of solid geometry, any of the definitions, axioms, and propositions of plane geometry may be used, as well as the definitions and axioms of solid geometry. It is therefore necessary to find out what planes exist in a figure, and what points and lines are in those planes, in order to be able to apply plane geometry to as great an extent as possible, and to guard against using plane geometry propositions where they do not hold.

9. Determining a Plane. To determine a plane, two facts must be proved: that the determining points or lines are such that they lie wholly in that plane, and that no other plane can contain them. It has already been said that a plane is determined by

(1) *three points not in a straight line;*

It is also determined by

(2) *A line and a point outside that line* ;

for two points on the line and the point outside determine the plane, and the line is wholly in the plane since two of its points are in the plane ;

(3) *two intersecting lines* (Prove as in (2)) ;

(4) *two parallel lines* (See the definition of parallel lines).

10. Representation of a Plane. Although a plane is unlimited in extent, only a limited part can be shown in the drawing of a figure, — just as in plane geometry only a limited part of an unlimited line is shown. A parallelogram has been found to be, in general, the best representation to use in such a drawing. It is well to make the side sup-

posed to be nearer the observer a little heavier than the others. Such a plane is denoted by a single letter, as plane *M*, by the letters at opposite vertices of the parallelogram, as plane *AB*, or by some determining points, or lines, as plane *ABC*, if it contains points *A*, *B*, and *C*, or plane *a*, *b* if it contains lines *a* and *b*. For a more extended discussion of figures, see Section III of the Preliminary Chapter.

EXERCISES

1. Does a triangle determine a plane? Why?
2. Why does a three-legged stool stand firmly on the floor, while a four-legged chair or table will sometimes rock?
3. If four points are given, how many planes can be determined? What is the smallest number of planes in which the sides of a quadrilateral can lie? In how many planes can all the sides of a parallelogram lie?

4. In a series of parallels such that not more than two lie in any one plane, how many planes would be determined by using three of them? four?

5. Show that all transversals of two intersecting lines are in the same plane.

11. Coplanar Figures. Figures in the same plane—such as two intersecting lines—are said to be *coplanar*, those not in the same plane are said to be *non-coplanar*. When figures are said to lie in the same plane, or to be coplanar, it is meant that a plane *can be passed through the figures* so as to entirely contain them,—not that there is necessarily a *given* plane that does contain them. Similarly, when any figures are said to be non-coplanar, the meaning is that *no plane could be passed so as to contain them*. The fact that two lines, or other parts, appear in different planes of a given figure does not, therefore, mean that they must be non-coplanar, for there may be some other plane that can be passed through them.

SECTION II. RELATIVE POSITIONS OF LINES AND PLANES IN SPACE; PARALLELS AND INTERSECTIONS

12. Necessity of the Study of Relative Positions. In plane geometry the study of the relative positions of points and lines is very informal, because the possibilities are few; but in solid geometry, points, lines, and planes admit of many relative positions. In order to give a clear idea of the figures, their more fundamental combinations will be studied.

13. Relative Positions of Two Lines. Two lines can be

(1) **Coplanar**, therefore, as in plane geometry;

(a) intersecting.

(b) parallel.

(2) **Non-coplanar**, therefore neither intersecting nor parallel.

An illustration of non-coplanar lines can be shown by crossing two pencils, and then moving one away from the other without changing the position of either except for this separation. The edge of the side wall and the ceiling of a room, and the edge of the back wall and the floor would also be a pair of such lines.

The *angle between two non-coplanar lines* means the angle between one of them and an intersecting line parallel to the other. All such angles formed for a certain pair of lines are equal, as is proved in § 42.

When two lines are neither intersecting nor parallel, they are not coplanar. When two lines are given without

any conditions that show without further proof whether or not they are intersecting or parallel, they cannot be assumed to determine a plane. One of them with a point of the other does, however, determine a plane, and if the second line should also lie in that plane, it may then be possible to prove this fact.

6. Transversals of two non-coplanar lines are neither parallel nor concurrent unless they meet at a point on one of the lines.

14. Relative Positions of a Line and a Plane. A line either (1) lies in a given plane; (2) intersects the plane; or (3) is parallel to the plane.

A line has already been said to *lie in a certain plane* if two of its points are in that plane. Other methods of proving that it lies in a certain plane will be taken up later.

A line *intersects a plane* if it meets the plane in one point only. The point is called the *foot* of the line with respect to that plane.

A line *is parallel to a plane*, and the plane is parallel to the line, if they have no point in common. The given line evidently cannot meet any line in the plane, but it is parallel only to such of those lines as are coplanar with it. Why?

15. Relative Positions of Two Planes. Two planes either (1) coincide; (2) intersect; or (3) are parallel. The study of two planes will also include some use of a line with the two planes.

16. Planes *coincide* if no point of one is outside the other. Coincidence is proved by showing the two planes to be determined by the same determining points, lines, or points and lines. (See § 9.)

17. Two planes *intersect* when they have at least one common point, but do not coincide.

18. Axiom on Intersecting Planes. *Two intersecting planes have at least two points in common.*

19. Theorem I. *The intersection of two planes is a straight line.*

They intersect in two points. What is known of the line through those points? Can the two planes have any other points in common?

7. Three planes that do not contain the same straight line can have but one point in common.

8. There can be but one line from an external point cutting two non-coplanar lines.

9. If a paper is folded and creased smoothly, why does it form a straight edge along the fold?

20. COR. *If a line is parallel to a plane, it is parallel to the intersection of any plane in which it lies, with that plane.*

10. If a line in one of two intersecting planes is parallel to the other plane, it is parallel to the intersection.

11. If a line is parallel to a plane, any two parallels drawn from that line to the plane are equal.

21. Parallel Planes. Two planes are parallel when they have no points in common. If two planes are parallel, either plane is, by definition, parallel to all the lines in the other; also, since any line in one of the planes is parallel to the other plane, it cannot meet any line in that other plane, but is parallel only to such of those lines as are coplanar with it. Why?

12. What is known about the relative positions of two lines that are parallel to the same plane?

22. Proving a Line to lie in a Plane. It has already been said that a line is in a plane if the plane contains two of its points. There are at present three other ways in which a line can be shown to be in a plane.

** 23. A line is in a plane if the plane contains one of its points, and also contains a second line to which the given line is parallel.*

Compare the plane determined by the parallels, and the given plane.

A direct consequence of this statement is

** 24. A plane containing one, and but one, of two parallels is parallel to the other.*

For if the second parallel should have a point in the plane, what would follow?

When this statement is used to show that a line is parallel to a plane, the following wording is convenient: *If a line outside a given plane is parallel to a line in the plane, it is parallel to the plane.*

** 25. A line is in a given plane if the plane contains one of its points, and if the line and the plane are both parallel to a second line.*

The parallel lines determine a plane; examine its intersection with the given plane.

** 26. A line is in a given plane if the plane contains one of its points, and if the line and the plane are both parallel to a second plane. Use § 25.*

27. The four methods, thus far considered, of proving a line in a plane, might be summed up as follows:

A line is in a given plane if one of its points is in that plane, and

- (a) if a second point of the line is in the plane;*
- (b) if a second line parallel to the given line is in the plane;*
- (c) if the line and the plane are both parallel to the same line;*
- (d) if the line and the plane are both parallel to the same plane.*

In each case two conditions are necessary, one being that a point of the given line shall be in the given plane.

13. If a line and a plane are parallel to the same line, they are parallel to each other, or the line lies in the plane.

14. If a line and a plane are parallel to the same plane, they are parallel to each other, or the line lies in the plane.

28. Transversals of Parallels. The following propositions concerning transversals of parallel lines and planes are direct consequences of § 27.

* (1) *If a plane cuts one of two parallel lines, it cuts the other also.* For if not, it is parallel to it, and so must contain the first line.

* (2) *If a line cuts one of two parallel planes, it cuts the other also.* For if not, what follows?

For a third case, dealing with three planes, see § 36.

15. To construct a line through a given point parallel to a given line.

16. To construct a line through a given point parallel to a given plane.

17. There can be an unlimited number of lines through a point outside a plane parallel to that plane.

29. Theorem II. *Two intersecting lines parallel to a plane determine a plane parallel to that plane.*

For if the second plane met the given plane, the two intersecting lines would both be parallel to the line of intersection of the two planes. Why?

30. COR. *Through any point there can be passed a plane parallel to any given plane not containing that point.*

31. To Construct a Plane. A plane is said to be constructed when its determining elements have been constructed. If two intersecting lines have been drawn, a plane can be said to be passed through them, that is, to be drawn so as to entirely contain them, and the plane is then considered as having been constructed, and can be used in any further construction.

It is obvious that the passing of the plane itself does not necessarily require the drawing of any additional lines in the figure. Construction in solid geometry differs from that in plane geometry in that it cannot actually accomplish all the construction in the plane of the drawing, and so must explain how to draw the determining elements for all the planes required, and how to construct whatever is to be drawn in those planes.

The figure drawn for a construction represents the three-dimensional figure as it should look to the observer, instead of showing each part in its true relation to the other parts. For example, lines that are to be constructed parallel to each other, or perpendicular to each other, cannot always be constructed in the plane of the drawing so as actually to have those relations. The student should explain what needs to be done in working in three dimensions, and should draw a figure to represent the result as nearly as possible. Reread Preliminary Chapter, Section I, B, VI.

32. CONSTRUCTION I. *Through a given point to construct a plane parallel to a given plane.*

18. Use § 27 and § 29 to prove Theorem V.

19. Through a given point to draw a line parallel to one of two given intersecting planes and meeting the second given plane at a given distance from the point.

33. Relative Positions of Three Planes. Three planes can intersect in (1) three lines; (2) two lines; (3) one line; (4) no lines,—that is, be mutually parallel, see § 37. Why can they not intersect in more than three lines?

34. Three Planes Intersecting in Three Lines. Each plane must intersect each of the other planes.

Theorem III. *If three planes intersect in three lines, the lines of intersection are (1) concurrent, or (2) parallel.*

Call the planes A , B , C , and the lines of intersection, according to the planes in which they lie, ab , bc , ca .

(1) If ab and bc meet in point P , in how many of the planes does P lie? Is it therefore in ca ? Why? Note that this proves that the lines of intersection are concurrent if any two of them meet.

(2) If ab and bc do not meet, they are parallel. Why?

Then can any two of the lines meet, by the conclusion of (1)? What follows?

35. Three Planes Intersecting in Two Lines. Two of the planes must be parallel, with the third plane intersecting each of the parallel planes. Why?

Theorem IV. *The intersections of two parallel planes with a third plane are parallel to each other.*

20. The sects cut off on parallel lines by parallel planes are equal.

36. Theorem V. *Through a given point not more than one plane can be passed parallel to a given plane.*

Suppose two planes through a point parallel to a third plane, and examine the intersections of these three planes with another plane drawn through the given point, but not containing the line of intersection of the two planes through that point. Why should the plane be drawn in this way?

This theorem might be worded :

Two intersecting planes cannot both be parallel to the same plane, or, as a theorem on transversals, A plane that intersects one of two parallel planes must intersect the other also.

It follows from this theorem that

37. COR. *Planes parallel to the same plane are parallel to each other.*

38. Three Planes Meeting in One Line. It is evident that any number of planes can be passed through one line. The leaves of a book roughly illustrate this possibility.

39. Three Planes Mutually Parallel. That this is possible was shown in § 37.

40. Theorem VI. *If two lines are parallel to a third line, they are parallel to each other.*

For what case is this already known? What is the new case?

FIRST METHOD

Let lines R and S be parallel to line C . Then there are determined planes RC and SC . (Why?) S and a point P in R determine a third plane. Examine the intersections of these three planes, and so show that R is parallel to S .

SECOND METHOD

Pass a plane M through S and a point P in R ; then,
 (1) M is parallel to C (why?), and therefore contains R . But R cannot meet S , because if it did R and S would be two intersecting lines parallel to C .

THIRD METHOD

(2) If M cuts R , it also cuts C (why?); also S , etc.

21. Two planes each passed through one of two parallel lines, are either parallel or meet in a line parallel to the given lines.

22. If two planes are parallel to the same line, they are parallel to each other or intersect in a line parallel to the given line.

23. As a door is opened, show that all the positions of its outer edge are parallel to each other.

24. Describe the figure of three parallel lines, without using the word "parallel."

25. If two planes are parallel to the same line, their intersections with any plane through that line are parallel to each other.

26. If a line is parallel to a plane, the intersections of that plane with all the planes through the given line are parallel to each other.

41. Lines or Planes Parallel to the Same Line or Plane.

All possibilities of lines and planes that are parallel to the same line or plane have now been considered, with the following results :

(1) Parallel to the Same Line.

- (a) Two lines ; they are parallel to each other.
- (b) A line and a plane ; they are parallel to each other, or the line lies in the plane.
- (c) Two planes ; it is not known whether they intersect or not. If they do meet, their line of intersection is parallel to the given line.

(2) Parallel to the Same Plane.

- (a) Two planes ; they are parallel to each other.
- (b) A line and a plane ; they are parallel to each other, or the line lies in the plane.
- (c) Two lines ; it is not known whether they are intersecting, parallel, or non-coplanar. If they meet, their plane is parallel to the given plane.

To sum up: If the three are alike, — that is, all lines or all planes, — they are all parallel.

If two unlike are parallel to a third, the two are parallel, or the line lies in the plane. It appears that a line being parallel to a plane, and a line being in the plane, are usually special cases of one general proposition.

If two alike are parallel to a third unlike, the relative position of the two is not fixed.

42. Theorem VII. *If two intersecting lines in one plane are respectively parallel to two intersecting lines in a second plane,*

(1) *the two planes are parallel;*

(2) *the angles in one plane are respectively equal to or supplemental to the angles in the other plane (according to their relative positions with regard to the vertices).*

For (1), see Theorem II.

(2) What method of proving angles equal can be applied even if the angles are in different planes? Cut off equal lengths on the arms of a correspondingly placed pair of angles, find what planes are determined, and see whether the material necessary for this method can be obtained.

43. The Possibility of passing Planes through One Line Parallel to a Second Line.

(1) If the lines intersect, no plane can be drawn through one line parallel to the other. Why?

(2) If the lines are parallel, any plane through one of the lines (except the plane determined by the two lines) is parallel to the other line. Why?

(3) If the lines are non-coplanar, then Theorem VIII follows.

44. Theorem VIII. *If two lines are non-coplanar, one and but one plane can be passed through either parallel to the other.*

From any point in one, draw a line parallel to the other. How can this be done, why is it done, and what conclusion follows?

Also, this is the only plane possible, for a plane through the first given line parallel to the other given line must contain the line constructed. Why?

45. CONST. II. *Through one of two given non-coplanar lines to draw a plane parallel to the other.*

46. Theorem IX. *Through a given point, one and but one plane can be passed parallel to two non-coplanar lines.*

Follow the same method of proof as in Theorem VIII.

In what special case will the plane contain one of the lines instead of being parallel to it? Note the similarity to the conclusion in § 41, 1, (b) and 2, (b).

47. CONST. III. *Through a given point to construct a plane parallel to two given non-coplanar lines.*

48. Theorem X. *If two lines are cut by three parallel planes, their corresponding sects are proportional.*

If the lines are parallel or concurrent, this has been proved in plane geometry. Explain why. What is the third case? In order to determine planes (why are they needed?), draw a third line so that it will determine a plane with each of the given lines, then use plane geometry. Find two ways to draw this auxiliary line.

49. COR. *If lines from a point (sometimes called a sheaf of lines) are cut by parallel planes, the corresponding sects on any two lines are proportional.*

27. The planes of a biplane are six feet apart, and two twelve-foot braces pass through the planes and to a point four feet below the lower plane. What length is cut off on them by the planes?

28. If two congruent parallelograms lie in parallel planes with their sides respectively parallel, how many planes are determined by using their sides in all possible combinations?

29. How many intersections have the planes determined in Ex. 28, and what can be told of these intersections? Do any of these intersections determine other planes? Examine the intersections of these planes with each other and with the original planes.

SECTION III. LINES PERPENDICULAR TO PLANES

(At the discretion of the teacher §§ 50 and 51 may be omitted, § 53 being assumed.)

*** 50.** *The Shortest Line from a Point to a Plane. From any given external point to a plane, there is one sect shorter than any other.*

The sects from P to plane M are of various lengths, for if a line is drawn in M the sects from P to points on that line are of various lengths. (Why?) Therefore there is some one length that is the shortest possible length from P to M .

No two sects, as PA and PB , could both be of this shortest length, for they determine a plane meeting M in AB , and being equal, both are longer than the perpendicular from P to AB . (Explain why.)

Therefore but one sect has the shortest possible length from P to M , that is, there is one sect from P to M that is shorter than any other.

*** 51.** *The shortest sect from a given external point to a given plane is perpendicular to all lines in the plane through its foot.*

For if there were a line through its foot to which it was not perpendicular, the perpendicular from the given point to that line would be shorter, which is not possible.

52. Line Perpendicular to a Plane. A line is *perpendicular to a plane* (and the plane is *perpendicular to the line*) when the line meets the plane and is perpendicular to all lines in the plane through its foot. When a line meets a plane and is not perpendicular to it, it is *oblique* to the plane. The term *normal* is sometimes used instead of *perpendicular*.

53. Existence of a Line Perpendicular to a Plane, and of a Plane Perpendicular to a Line. In §§ 50 and 51, it was proved that

From any given external point to any given plane, there is a line perpendicular to that plane.

It also follows that

At any given point of a given plane, there is a line perpendicular to that plane, and

At any given point in a given line, there is a plane perpendicular to that line.

For, since there can be a line perpendicular to a plane, that figure can be superposed so as to make the line perpendicular to any plane at any point in it, or so as to make the plane perpendicular to any line at any point in it. Explain how this can be done.

To sum up, it is possible to assume in a figure:

- (1) A line perpendicular to any plane
 - (a) from any external point;
 - (b) at any point in the plane.
- (2) A plane perpendicular to any line
 - (a) at any point in the line.

(b) It has not yet been shown, but will follow from §§ 62 and 63, that there can be a plane perpendicular to any line from any external point.

54. Theorem XI. *Two intersecting lines cannot be perpendicular to the same plane.*

The point of intersection can be either in or outside the plane. Suppose two perpendiculars to the given plane through this point; they determine a plane, and have what relation to the intersection of the two planes?

55. Theorem XII. *Two intersecting planes cannot both be perpendicular to the same line.*

Pass a plane through the given line so as to meet the two intersecting planes in intersecting lines. Can these lines both be perpendicular to the given line?

It follows that

56. COR. *Planes perpendicular to the same line are parallel.*

57. Theorem XIII. *A line perpendicular to one of two parallel planes is perpendicular to the other also.*

For it meets the second plane (why?), and by the use of parallels it can be shown to be perpendicular to any line in that plane. Or,

Draw a plane perpendicular to the line at the second intersection and prove that it coincides with the given plane.

30. Planes perpendicular to intersecting lines are not parallel.

58. Theorem XIV. *If one of two parallel lines is perpendicular to a plane, the other also is perpendicular to that plane.*

It meets the plane (why?); then use parallels to prove it perpendicular to every line through its foot.

31. If an oblique line cuts two parallel planes, the angles it makes with the lines through its foot in one plane are equal to the corresponding angles it makes with the lines through its foot in the other plane. What determines which are the corresponding angles in the two planes?

59. Theorem XV. *Lines perpendicular to the same plane are parallel.*

Take any two, and through the foot of one draw a parallel to the other. What follows?

32. All perpendiculars from a given line to a given plane are coplanar. In what positions might the given line lie?

33. Sects of perpendiculars between parallel planes are equal.

60. Theorem XVI. *Any line perpendicular to a given line at a given point lies in the plane perpendicular to the line at that point.*

Pass a plane through the two lines, and examine the intersections of the two planes.

Note that this is a fifth method of showing that a line is in a plane. It might be stated as follows:

A line is in a plane if it has one point in the plane, and if the line and the plane are both perpendicular to the same line at the same point. Show that if the last four words are omitted, the statement is still true.

61. COR. 1. *All the perpendiculars that can be drawn to a given line at a given point are coplanar.*

62. COR. 2. *A line perpendicular to two intersecting lines is perpendicular to their plane.* For they lie in and determine the perpendicular plane.

34. Two lines perpendicular to the same line at the same point determine a plane perpendicular to the line at that point.

35. One arm of a right angle revolved about the other arm as an axis generates a plane. Does the arm of any other angle generate a plane?

36. The hand of a clock revolves in a plane. Does the arm of a hoisting crane revolve in a plane?

63. CONST. IV. *To draw a plane through a given point perpendicular to a given line.*

In how many positions might the point be? How much is needed to determine the required plane?

SECTION IV. ANGLES BETWEEN PLANES

64. Dihedral Angles. An angle between two intersecting planes is called a *dihedral angle*. The planes are called the *faces*, and their intersection is called the *edge* of the dihedral angle.

Two intersecting planes evidently form four dihedral angles, just as two intersecting lines form four plane angles. In the figure, the planes M and N , intersecting in XY , form the dihedral angles indicated by the arrows as A , B , C , and D . A dihedral angle may be denoted by naming its faces, with or without the use of the intersection, as dihedral angle MN , or dihedral angle $M-XY-N$. When, as in this figure, there might arise confusion as to which angle was intended, a second letter may be used for each plane, as M' and N' in the figure, and their order in the naming of the angle will determine which angle is meant. Thus, angle C could be named $M'-XY-N'$, the direction of rotation being considered, as in plane geometry, counterclockwise (that is, in the direction opposite to that in which the hands of a clock rotate).

As the size of a plane angle does not depend on the length of its arms, so the size of a dihedral angle does not depend on the extent of its faces, but only on the amount of rotation necessary for a plane to rotate about the intersection from one face to the other.

Familiar illustrations of dihedral angles are, the angle between two leaves of an open book, the angle between two walls of a room, the angle between two planes of a roof, the various angles made as a desk cover is raised or as a door is opened, etc.

65. Plane, or Measuring, Angle. If at a point in the edge of a dihedral angle, two perpendiculars, one in each face, are drawn to the edge, the angle between the perpendiculars is called the *plane angle*, or the *measuring angle* of the dihedral angle. This plane angle evidently lies in, and its arms determine, a plane perpendicular to the edge.

It is evident that all plane angles of the same dihedral angle are equal. Why?

66. Theorem XVII. *Plane angles of equal dihedral angles are equal.*

Superpose.

37. The greater of two unequal dihedral angles has the greater plane angle.

67. Theorem XVIII. *Two dihedral angles are equal if their plane angles are equal.*

38. Vertical dihedral angles are equal. (See def. in § 70.)

68. Theorem XIX. *Two dihedral angles are proportional to their plane angles.*

Assume a common divisor for the commensurable case.

69. Measurement of a Dihedral Angle. Since the amount of rotation in a dihedral angle is the same part of a complete rotation about its edge that its plane angle is of a perigon (Th. XIX), the plane angle is the measure of the dihedral angle, this complete rotation being considered as a standard by which to measure. It is on this account that the plane angle is called *the measuring angle of the dihedral angle*.

It is evident, therefore, that propositions concerning dihedral angles will usually be proved by the use of their measuring angles.

70. Terms used for Plane Angles that apply also to Dihedral Angles. On account of the dependence of dihedral angles on their plane angles, a dihedral angle is called *acute, right, obtuse, straight, or reflex*, according as its plane angle is acute, right, obtuse, straight, or reflex; also two dihedral angles are *adjacent, vertical, complementary, supplementary, or explementary*, according as their measuring angles fulfill the conditions of the definitions of these terms.

If the dihedral angle between two planes is a right dihedral angle, the planes are said to be *perpendicular* to each other.

71. Theorem XX. *If a line is perpendicular to a plane, any plane through that line is perpendicular to the given plane.*

How can planes be proved perpendicular? What auxiliary line does this necessitate?

Note that this is a second method of proving planes perpendicular, and that it is simpler to apply than the definition, since that requires the use of two lines, while this method requires but one.

39. How many intersecting planes can be perpendicular to the same plane?

72. Theorem XXI. *A line drawn in one of two perpendicular planes, perpendicular to their intersection, is perpendicular to the other plane.*

This proves that each of two perpendicular planes contains an unlimited number of lines perpendicular to the other plane. Are all the lines in one plane perpendicular to the other? Are all the lines that meet the intersection perpendicular to the other plane?

40. If a line and a plane intersect, a plane perpendicular to one is not perpendicular to the other.

73. COR. 1. *If two planes are perpendicular, a line perpendicular to one of them through a point in the other, lies wholly in that other.*

Use the theorem.

Stated like the other methods of proving a line in a plane, this becomes: *A line is in a plane if it has one point in that plane, and if the line and the plane are both perpendicular to the same plane.*

74. COR. 2. *If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to that plane.*

What is known of a perpendicular to the third plane through any point of the given intersection?

41. Can planes perpendicular to intersecting planes be parallel? Explain.

75. COR. 3. *If two intersecting lines are perpendicular to the faces of a dihedral angle, they determine a plane that contains one of the measuring angles of that dihedral angle.*

76. Theorem XXII. *Through any line not perpendicular to a given plane, one, and but one, plane can be passed perpendicular to that plane.*

Use the simplest method of determining a plane perpendicular to the given plane, then show that a perpendicular plane must contain these determining lines.

77. Projections. The *projection of a point upon a plane* is the foot of the perpendicular from that point to the plane. The *projection of a line upon a plane* is the locus of the projections of all its points upon that plane.

78. Theorem XXIII. *The projection upon a plane of a line that is not perpendicular to that plane is a straight line.*

Prove (1) the projections of all its points will lie in one line; (2) any point in that line is the projection of some point of the given line. What would be the projection of a perpendicular upon a plane?

NOTE. The following group of exercises on projections is closely associated with the elements of *descriptive geometry*.

Given two planes perpendicular to each other:

42. Find the line of which two given lines, one in each plane (but not both perpendicular to the intersection) are the projections upon those planes.

43. If a line is perpendicular to one of the planes, its projection upon the other is perpendicular to the intersection.

44. If a line is parallel to one of the planes, its projection upon the other plane is parallel to the intersection.

45. If a line is parallel to both planes, its projections upon them are parallel to each other.

46. If two intersecting lines are projected upon both planes, the line joining the intersection of their projections upon one plane with

the intersection of their projections upon the other plane is in a plane perpendicular to the intersection of the given planes.

47. If a third plane is perpendicular to either of the given planes, its intersection with the other plane is perpendicular to the intersection of the given planes.

48. If a given line is perpendicular to a third plane, its projections upon the given planes are respectively perpendicular to the intersections of the third plane with the given planes.

49. The projection of any line in one upon the other is the intersection.

79. Theorem XXIV. *Of all lines drawn from a given external point to a given plane,*

(a) *The perpendicular is the shortest.*

(b) *Those making equal angles with the perpendicular, or having equal projections on the plane, are equal.*

(c) *Those making unequal angles with the perpendicular, or having unequal projections on the plane, are unequal, the one making the greater angle, or having the longer projection, being the longer.*

Use plane geometry.

80. Distance from a Point to a Plane. The length of the perpendicular from a point to a plane is called the *distance* from the point to the plane. Why?

81. COR. *Equal obliques from a point to a plane make equal angles with the perpendicular, and have equal projections upon the plane.*

Unequal obliques from a point to a plane make unequal angles with the perpendicular, and have unequal projections upon the plane, the longer oblique making the greater angle and having the longer projection.

What relation has this to (b) and (c) of the theorem?

50. All points in a given plane at a fixed distance from a given external point lie in a circle. All points in a circumference are equidistant from any point on the perpendicular to their plane at the center of the circle.

51. From any point not in the perpendicular at the center of a circle, only two points in the circumference are equidistant. How could the second of such a pair of points be found if the first is given?

82. Theorem XXV. *The acute angle that an oblique makes with its projection upon a plane is the least angle it makes with any line in the plane.*

How can angles not in the same triangle be compared in size? Use a perpendicular from the given line to its projection.

83. The Angle between a Line and a Plane. The lesser of the two angles that a line makes with its projection upon a plane is called its *angle with the plane*, or its *inclination to the plane*. Why is this angle used?

52. Which is the greatest angle that an oblique line makes with any line in a given plane?

53. Parallel lines are equally inclined to a plane.

54. A line is equally inclined to two parallel planes.

55. If one of two parallels meets a plane, and the other meets a parallel plane, their inclinations to those planes are equal.

56. Equal obliques from a point to a plane make equal angles with that plane.

84. Theorem XXVI. *If a line meets its projection upon a plane, the line in the plane perpendicular to one of them at their intersection is perpendicular to the other also.*

The line and its projection determine a plane. Prove the two planes perpendicular to each other. Or,

Cut off equal sects on the perpendicular, and then use obliques. This proof is practically the same, although in reversed order, in the two cases.

57. A line oblique to a plane is concurrent with two lines in the plane, and its projection bisects the angle between them. Prove that it makes equal angles with those lines.

58. Any line in one of two perpendicular planes, meeting the intersection, is perpendicular to an arm of a measuring angle.

59. If from a point outside a plane, perpendiculars are drawn to the arms of a right angle in that plane, they include an acute angle.

60. If from a point outside a plane, perpendiculars are drawn to the arms of an angle in that plane, they include an angle less than its supplement.

61. If a figure has four straight line sides and four right angles, it lies in one plane, and is a rectangle.

62. If from a point in one of two oblique planes a perpendicular is drawn to the intersection, and a second perpendicular is drawn to the other plane, the plane determined by those lines is perpendicular to both the given planes and to their intersection.

85. CONST. V. *To draw a plane through a given point perpendicular to a given plane.*

Draw any line in the plane, then draw a plane through the given point perpendicular to that line. Why are the planes perpendicular?

86. CONST. VI. *To draw a line through a given point perpendicular to a given plane.*

First draw a plane through the point perpendicular to the given plane. How can a line be drawn in this plane so as to be perpendicular to the given plane?

How can this be used to construct a plane through a given line perpendicular to a given plane?

63. To construct a plane through a given point so as to make equal angles with two given intersecting planes.

87. Theorem XXVII. *Two non-coplanar lines have one, and but one, common perpendicular, and it is the shortest line that can be drawn between them.*

Pass a plane through each of the lines parallel to the other, and a plane through each of the lines perpendicular to these parallel planes. Where is the common perpendicular? Show that it is the only one possible. The pupil may be able to prove this theorem equally well with but two of these planes drawn.

The length of their common perpendicular is called the *distance* between two non-coplanar lines. Why?

64. If a line is parallel to a plane, its distances from all the lines in the plane that are non-coplanar with it are equal.

SECTION V. LOCUS OF POINTS

88. Loci. The proof of a locus theorem must, as in plane geometry, consist of two parts, — the proof that

(a) Every point that fulfills the given conditions lies on a certain figure (usually consisting of one or more points, lines, or surfaces); and

(b) Every point on that figure fulfills the conditions.

The negative converse of either of these statements can be proved in its stead if that seems desirable, but such a possibility is not likely to arise.

One locus theorem, Theorem XXIII, has already been demonstrated, but as it is used to show that a projection is straight, rather than to find a locus, it is not classed with the theorems of this section.

65. Find the locus of points at a fixed distance from a given plane.

66. Find the locus of points at fixed distances from each of two given planes.

89. Theorem XXVIII. *The locus of points equally distant from two points is the plane perpendicular to the line joining them, at its middle point.*

Use plane geometry, with what is known about a plane that is perpendicular to a line, to prove that

(a) Any point equidistant from the two points is in that plane.

(b) Any point in that plane is equidistant from the two points.

Note that this proves also that

(c) Any point not in the plane is not equidistant from the two points.

(d) Any point not equidistant from the two points is not in the plane.

90. Theorem XXIX. *The locus of points equidistant from three non-collinear points is the perpendicular to their plane at the circumcenter of the triangle formed.*

Prove it by Theorem XXVIII, applied twice, or use projections.

91. Theorem XXX. *The locus of points equidistant from four non-coplanar points, no three of which are collinear, is a point.*

FIRST METHOD

Let the points be A , B , C , and D . Erect the perpendicular OR to plane ABC at the circumcenter O , and draw plane K , the perpendicular bisector of AD . Since AD intersects plane ABC (why?), K is not perpendicular to ABC (why?) and meets OR at some point P . What is known of P ?

SECOND METHOD

Draw AB , AC , and AD , and their perpendicular bisecting planes L , M , and N . Then no two of these planes are parallel. (Why?) They intersect either (a) in one line, (b) in three parallel lines, or (c) in three concurrent lines, and, therefore, all in a point. But the intersection of L and M would be perpendicular to the plane ABC , and the intersection of M and N would be perpendicular to the plane ACD , therefore (a) and (b) are impossible, and (c) is the only possibility.

THIRD METHOD

Draw triangles ABC and BCD ; erect perpendiculars to their planes at the circumcenters of those triangles, and prove that these perpendiculars lie in a plane and intersect. This is probably the hardest method of the three, but all are good practice in loci.

Note that if the six sects joining the four points are drawn, each plane that is the perpendicular bisector of one of these sects must contain the point equidistant from all the points; therefore these six planes have intersections that are concurrent at this point.

67. Find the points in a given plane that are equidistant from two given points.

68. Find the locus of points in a given plane, equidistant from three points not in a line.

69. Find the locus of points equidistant from two lines (a) parallel, (b) intersecting. Remember that distance from a point to a line always means distance along a perpendicular.

70. Find the locus of points equidistant from three concurrent lines.

92. Theorem XXXI. *The locus of points equidistant from two intersecting planes is the pair of planes that bisect their dihedral angles.*

(1) **To prove.** A point equidistant from two intersecting planes lies in a plane bisecting the dihedral angle between whose faces the point lies.

Let the planes M and N meet in XY , and let point P be equidistant from M and N , that is, let the perpendiculars PA and PC to those planes be equal. Plane B is determined by P and XY ; it is then necessary to prove that B bisects the dihedral angle MN . PA and PC determine the plane K of the measuring angle ARC . (Why?) Is P equidistant from the arms of the measuring angle ARC ? (Why?) Where then must P lie?

(2) Prove the converse, using the same method of construction.

71. Find the locus of points equidistant from two parallel planes.

72. Where do the midpoints of all transversals of two non-coplanar lines lie? Is this therefore a locus?

NOTE. The locus of points equidistant from two points, from two coplanar lines, and from two planes has been found. It is not possible, however, in elementary geometry, to find the locus of points equidistant from two different elements, as a point and a line, a line and a plane, a point and a plane, or from two non-coplanar lines. Such loci are discussed in analytical geometry and related subjects.

73. If three planes meet in one point, show that the bisectors of the dihedral angles formed are concurrent in a line.

74. If four planes meet, each three in a point, show that the bisectors of the six dihedral angles meet in a point.

93. COR. 1. *To find the locus of points equidistant from three planes that meet in one point.*

94. COR. 2. *To find the locus of points equidistant from four planes, each three of which meet in one point.*

75. If three planes intersect in three concurrent lines, show that the planes perpendicular to the given planes through the bisectors of the plane angles between their lines of intersection meet in a line.

76. Find a point in one of two given non-coplanar lines equidistant from two given points in the other.

77. Find the locus of points in a given plane equidistant from two given coplanar lines outside the plane.

78. Find the locus of a point such that the sum of the squares of its distances from two given fixed points shall be constant.

79. Find the locus of a point such that the difference of the squares of its distances from two given fixed points shall be constant.

80. Find the locus of points equidistant from two given planes and from two given points; from two given coplanar lines and from two given points; from two given coplanar lines and from two planes.

81. Find the locus of points at a fixed distance from a given plane and equidistant from two given points; from two given coplanar lines; from two given planes.

95.

SUMMARY OF PROPOSITIONS

(Numbers in parentheses refer to black-faced section numbers.)

I. THE STRAIGHT LINE:

(1) *Lines parallel:*

A line in one of two intersecting planes and parallel to the other is parallel to the intersection (20).

Two lines parallel to the same line are parallel to each other (40).

Two lines perpendicular to the same plane are parallel to each other (59).

The intersections of two parallel planes with a third plane are parallel to each other (35).

The intersections of three planes not all concurrent are parallel to each other (34).

(2) *Lines perpendicular :*

The shortest sect from a point to a plane is perpendicular to all lines in the plane through its foot (51); or, a line perpendicular to a plane is perpendicular to all lines in the plane through its foot (52).

A line in a plane and perpendicular either to a line or to its projection upon that plane at their point of intersection is perpendicular to the other also (84).

(3) *Lines concurrent :*

The intersections of three planes meeting in pairs are concurrent if any two of the intersections meet (34).

(4) *There is one, and but one, line*

through a given point perpendicular to a given plane (53, 54); perpendicular to two given non-coplanar lines (87).

(5) *To construct a line*

through a given point perpendicular to a given plane (86).

(6) *The projection*

upon a plane of a line oblique or parallel to that plane is a straight line (78).

(7) *Sects equal :*

Obliques making equal angles with the perpendicular or having equal projections upon the plane are equal (79).

Projections of equal obliques from a point are equal (81).

(8) *Sects unequal :*

From a given point to a given plane, one sect is the shortest (50).

The perpendicular from a point to a plane is shorter than any oblique from that point (79).

Obliques making unequal angles with the perpendicular, or having unequal projections upon the plane are unequal (79). Projections of unequal obliques from a point to a plane are unequal (81).

The common perpendicular is the shortest sect between two non-coplanar lines (87).

(9) *Sects are proportional :*

if they are cut by three parallel planes (48);

if they are from a given point, cut by two parallel planes (49).

II. THE PLANE:

(1) *Is determined*

by three non-collinear points (5);

by a line and a point outside (9);

by two intersecting lines (9);

by two parallel lines (9).

(2) *There is one, and but one, plane*

through a given point

parallel to a given plane (30, 36);

parallel to two given non-coplanar lines (46);

perpendicular to a given line (53);

through a given line

parallel to a given non-coplanar line (44);

perpendicular to a given plane (unless the given line is itself perpendicular to the plane) (76).

(3) *Two planes are parallel:*

when they have no point in common (21);

if one contains two intersecting lines parallel to the other (29);

if two intersecting lines of one are parallel respectively to two intersecting lines of the other (42);

if they are parallel to the same plane (37);

if they are perpendicular to the same line (56).

(4) *Two planes are perpendicular:*

when their measuring angle is a right angle (70);

if one contains a line perpendicular to the other (70).

(5) *To construct a plane*

through a given point

parallel to a given plane (32);

parallel to two given non-coplanar lines (47);

perpendicular to a given line (63);

perpendicular to a given plane (85);

through a given line

parallel to a given non-coplanar line (45).

III. LINES AND PLANES:

(1) *A line is in a plane:*

if two of its points are in the plane (6);

if one of its points and a line parallel to it are in the plane (23);

if one of its points is in the plane, and the line and the plane are both

parallel to the same line (25);

parallel to the same plane (26);

perpendicular to the same line (60, 61);

perpendicular to the same plane (73).

(2) *Relative positions*

of two lines: (a) intersecting, (b) parallel, (c) non-coplanar (13);

of a line and a plane: (a) the line in the plane, (b) intersecting, (c) parallel (14);

of two planes: (a) coinciding, (b) intersecting, (c) parallel (15);

of three planes: (a) intersecting in three lines, (b) intersecting in two lines, (c) all meeting in one line, (d) mutually parallel (33).

(3) *Intersections*

of a line and a plane, — a point (14);

of two planes, — a line (19);

of three planes:

a line (38);

two parallel lines (35);

three lines parallel or concurrent (34).

(4) *A line is parallel to a plane :*

- if they have no point in common (14);
- if it is not in the plane and is parallel to a line in the plane (24);
- if it is not in the plane, and the line and the plane are both parallel to the same line or plane (41).

(5) *A line is perpendicular to a plane :*

- if it is perpendicular to all lines in the plane through its foot (52);
- if it is perpendicular to two intersecting lines in the plane (62);
- if it is perpendicular to a plane that is parallel to that plane (57);
- if it is parallel to a line that is perpendicular to the plane (58);
- if it is in a plane perpendicular to it and is perpendicular to the intersection of the two planes (72);
- if it is the intersection of two planes that are perpendicular to that plane (74).

(6) *Transversals of parallels :*

- If a line cuts one of two parallel planes, it cuts the other also (28).
- If a plane cuts one of two parallel lines, it cuts the other also (28).
- If a plane cuts one of two parallel planes, it cuts the other also (36).

IV. ANGLES :

(1) *Equal plane angles :*

- Angles whose arms are respectively parallel are equal (42).
- Angles made by equal obliques with a perpendicular are equal (81).
- Measuring angles of equal dihedral angles are equal (66).

(2) *Unequal plane angles :*

- Angles made by unequal obliques with a perpendicular are unequal (81).
- The angle between an oblique line and its projection upon a plane is the least angle it makes with any line in the plane (82).

(3) *Dihedral angles*

having equal plane angles are equal (67);
 are proportional to their plane angles (68);
 are measured by their plane angles (69).

(4) *Measuring angles have their plane determined.*

by two lines in the faces perpendicular to the edge at the same point (65);
 by two intersecting lines perpendicular to the faces of the dihedral angle (75).

V. Locus:

of points equidistant from

two given points, — a plane perpendicular to their sect at its midpoint (89);

three given non-collinear points, — a line perpendicular to their plane at the circumcenter of the triangle of which they are vertices (90);

four given non-coplanar points of which no three are collinear, — the point of concurrence of the planes perpendicular to their sects at their midpoint (91);

two intersecting planes, — the planes bisecting the dihedral angles formed (92);

three planes meeting in a point, — the lines of intersection of the planes bisecting the dihedral angles (93);

four planes meeting in four points, — the points common to the bisectors of the dihedral angles (94).

96.

ORAL AND REVIEW QUESTIONS

What are the two most fundamental statements that can be made about a plane? In order to use plane geometry propositions in proving solid geometry theorems, what must first be done? By what methods can this be done? Show that two transversals of the arms of an angle are, in general, coplanar. What is the exception? In what relative positions can a point and a line lie? a point and a plane? a line and a plane? two lines? two planes? What is the intersection of a line and a plane? Why? of two planes? Why? State six methods of proving that a line is in a plane. Why is it ever necessary to prove that a line is in a plane? What is the definition of a line parallel to

a plane? Is there a simpler method of showing that a line is parallel to a plane? What is the definition of parallel planes? State two other methods of proving planes parallel, founded directly on this definition, and state and prove the other theorem used in each method. How many intersecting lines can be parallel to the same line? parallel to the same plane? How many intersecting planes can be parallel to the same plane? to the same line? State all possibilities for the relative positions of three planes. If three planes intersect in pairs, is it possible for them to have two non-coplanar intersections? Is it possible with four planes? Are planes necessarily parallel if they contain parallel lines? Are lines necessarily parallel if they are in parallel planes? How many lines through a point can be perpendicular to the same line? to the same plane? How many intersecting planes can be perpendicular to the same line? to the same plane?

Which of the following must be parallel (or coincident, including as a special case the line being in the plane): lines parallel to the same line; planes parallel to the same plane; lines parallel to the same plane; planes parallel to the same line; lines perpendicular to the same line; lines perpendicular to the same plane; planes perpendicular to the same line; planes perpendicular to the same plane? How many intersecting lines can be perpendicular to the same line? to the same plane? What is the definition of perpendicular planes? How many lines are used to show the planes perpendicular to each other? Can it be proved with a less number? If two planes are perpendicular to each other, how can a line be drawn in one so as to be perpendicular to the other? If two planes are perpendicular to each other, is every line in one perpendicular to the other? Are all perpendiculars to one plane at the intersection in the other plane? Where is the projection of lines in one plane on the other plane? What is the definition of a line perpendicular to a plane? How many lines of the plane does it use? Can the line be proved perpendicular to the plane with fewer lines? State two methods that use parallels to prove a line perpendicular to a plane. State two methods that use perpendicular planes to prove a line perpendicular to a plane.

Define dihedral angle, and its plane angle. In what two ways is the plane of a measuring angle determined? Explain in full in what sense and why the plane angle of a dihedral angle measures it. Can the projection of a straight line upon a plane ever be a curve? Can the projection of a curve ever be a straight line? Where must a pro-

jected line lie, in order that its projection upon a plane shall be a straight line (*a*) if it is straight? (*b*) if it is not straight? If a line when projected upon each of two planes makes straight lines on both, what must be true of it? (Two cases.) Give the determining lines that must be drawn in order to construct a plane (1) through a point (*a*) parallel to a given plane, (*b*) parallel to two non-coplanar lines, (*c*) perpendicular to a line, (*d*) perpendicular to a plane; (2) through a line (*a*) parallel to a non-coplanar line, (*b*) perpendicular to a plane. State two methods that have been used to prove two lines perpendicular. What methods have been found to prove angles equal? unequal? sects equal? unequal? sects proportional? Upon what plane geometry locus theorems can the following be made to depend: locus of points equidistant from two points? three points? two lines? two planes? State all the propositions that prove that there can be a line or a plane that fulfills certain conditions; all propositions that prove that there can be but one line or one plane that fulfills certain conditions.

GENERAL EXERCISES

82. If two perpendiculars to a plane from two points on the same side of it are equal, the line joining those points is parallel to the plane. If three or more such perpendiculars are equal, the points determine a plane parallel to the given plane.

83. If two planes are drawn perpendicular respectively to two non-coplanar lines, their intersection is perpendicular to any plane parallel to the given lines.

84. Find the locus of points in a given plane, equidistant from two given planes.

NOTE. Two points equidistant from a plane, on the same perpendicular to that plane, are called *symmetric* with regard to that plane.

85. Prove that two points symmetric to a plane are equidistant from any point in the plane.

86. If *A* and *B* are on the same side of plane *M*, find the shortest path between *A* and *B* that includes one point of *M*. (Use the point symmetric to either *A* or *B*.) This is simply finding the shortest way to go from *A* to *M* and back to *B*. It is the path that a ray of light travels when it meets a mirror and is reflected; it is the path followed by a billiard ball striking a cushion and bounding back, or

by a tennis ball bounding from the ground, — unless, in either of these cases, the ball is affected by some motion, such as whirling, other than the mere rebound.

87. If two balls are in positions A and B on a billiard table, show how the point on a cushion could be determined so that if A strikes the cushion at that point, it will rebound to B . Show how a point could be determined so that A would strike two cushions and rebound to B ; three cushions; all four cushions. Note that in the last cases symmetry is being used in regard to two or more planes.

88. Find the shortest path in two intersecting planes between a point in one of the planes, and a point in the other plane.

89. Find the shortest path in the surface of a box from one corner to the opposite corner.

90. If two lines in one of two intersecting planes make equal angles with the intersection, they make equal angles with the other plane, and conversely.

91. Perpendiculars from a point to a set of parallel lines are coplanar.

92. The arms of an angle are equally inclined to any plane through its bisector.

93. The arms of an angle are equally inclined to any plane through a coplanar line perpendicular to its bisector.

94. If a line is perpendicular to one of two intersecting planes, the projection upon the other plane is perpendicular to the intersection.

95. If any number of planes perpendicular to the same plane have a common point, they intersect in a common line.

96. If a plane and a line are parallel, a plane perpendicular to the line is also perpendicular to the plane. Is the converse also true?

97. If three or more lines are drawn from points on one of two non-coplanar lines perpendicular to the other line, the sects cut off on the two non-coplanar lines by these perpendiculars are proportional.

98. If a common perpendicular is drawn to two lines that are either parallel or non-coplanar, the plane perpendicular to that common perpendicular at its midpoint bisects every transversal of the two lines.

99. If two congruent rectangles are placed so that they have an edge in common but are not coplanar, their common edge is perpendicular to two of the planes determined by other edges, and is parallel to a third plane. State the general case.

100. When will lines perpendicular to two intersecting planes meet? Prove it.

101. Planes perpendicular to intersecting lines meet. What can be told about their intersection?

102. Given three lines, no two of which are coplanar: (a) pass a plane through one of them so as not to be parallel to either of the others; (b) draw a line through all three of them. Can a line be drawn through one of them parallel to the other two? a plane?

103. Given a line lying within and parallel to the edge of a dihedral angle, how could a plane be passed through this line, intersecting the faces of the angle, so that the parallels so formed would be equidistant from the given line?

104. Given a line outside a dihedral angle parallel to its edge, how could a plane be passed through this line meeting the faces of the dihedral angle so that the distance apart of the parallel intersections shall equal the distance of one of them from the given line.

105. If a circle is inscribed in a triangle, the lines from any point in the perpendicular erected to the plane of the circle at the center to the points of contact are perpendicular to the sides of the triangle.

106. The lines from any point in the perpendicular to a plane at the incenter of a given triangle, perpendicular to the sides of the triangle, meet them at the points where they are tangent to the inscribed circle.

107. If in each of two intersecting planes two lines are drawn parallel to the intersection, and such that all four lines are the same distance from the intersection, those lines determine two pairs of parallel planes.

108. If four lines determine more than four planes, the lines are either all concurrent or all parallel.

109. If two pairs of parallel planes intersect, the other two planes determined by their intersections meet in a line parallel to the given planes and to their intersections, and equidistant from each opposite pair of intersections.

110. If one pair of parallel planes is perpendicular to a second pair of parallel planes, the distance between one pair of opposite intersections equals the distance between the other pair of opposite intersections.

111. If three planes meet in parallel lines, and two of them make equal angles with the third, then the bisector of the dihedral angle between those two planes is perpendicular to the third plane.

112. If two pairs of parallel planes intersect, and the consecutive pairs of their intersections are the same distance apart, the diagonal planes of the figure formed are perpendicular to each other.

113. If two parallel planes are crossed by two non-parallel planes that make equal angles with one of them,

(a) those planes make equal angles with the other of the parallel planes;

(b) the opposite intersections are equidistant;

(c) the non-parallel planes meet in a line parallel to the given intersections, and equidistant from those in either of the parallel planes;

(d) the given intersections in one of the non-parallel planes are the same distance apart as the intersections in the other of those planes.

(e) State and prove the converse of (d).

114. If two triangles (as ABC and XYZ) in different planes are so placed that each corresponding pair of sides intersect (as, AB and XY , BC and YZ , CA and ZY), then the lines through their corresponding vertices (as AX , BY and CZ) are either parallel or concurrent.

BOOK VII. POLYHEDRONS, CYLINDERS, AND CONES

SECTION I. THE PRISM AND THE CYLINDER

97. Polyhedrons. A geometric solid bounded by planes is a *polyhedron*. The polygons that bound it are its *faces*, their intersections are its *edges*, and the intersections of its edges are its *vertices*; the faces taken together make up its *surface*. The area of its surface is the *area of the polyhedron*, and the amount of space that it occupies is the *volume of the polyhedron*. See also § 135, which explains how volume is measured.

A POLYHEDRON

A TETRAHEDRON

98. Number of Faces. It is evident that, in order to inclose space, a polyhedron must have at least three faces about each vertex, and at least four faces in all. A solid bounded by four faces is called a *tetrahedron*; by five faces, a *pentahedron*; by six faces, a *hexahedron*; by eight faces, an *octahedron*; by ten faces, a *decahedron*; by twelve faces, a *dodecahedron*; and by twenty faces, an *icosahedron*. The corresponding prefixes are used for the polyhedrons of any number of faces, but those here mentioned include the most common.

99. Polyhedral Angles. When three or more planes meet in a point, they include a *polyhedral angle*, or a *solid angle*.

In the figure, the planes OVW , OWX , OXY , OYZ , OZV meet in O , and form a polyhedral angle O , or more definitely $O-VWXYZ$.

The common point O is its *vertex*, the portions of planes that bound it are its *faces*, and the intersection lines of its faces are its *edges*. The faces and edges extend indefinitely from the vertex, but, for convenience in dealing with a polyhedral angle, a plane, as $ABCDE$, is sometimes considered as having cut its edges.

100. Dihedral and Face Angles. A polyhedral angle evidently has as many dihedral angles as edges, and as many *face angles*,—that is, angles about the vertex in its faces, as $\angle AOB$, $\angle BOC$, etc.,—as faces. The dihedral angles and face angles are sometimes called its *parts*, just as the sides and angles of a triangle are referred to as its parts.

A polyhedron has a polyhedral angle at each vertex, a dihedral angle at each edge, and face angles about each vertex, the number of face angles depending on the kind of polyhedral angle at that point.

101. Sections. When a plane is passed through a solid, the figure in the plane bounded by its intersection with

the surface of the solid is called a *plane section*, or simply a *section*, of the solid. It is evident that any section of a solid must be bounded by a closed line, or lines. Why?

Similarly, a plane passed through all the edges of a polyhedral angle, but not through the vertex, forms a polygon of as many sides as the polyhedral angle has faces. In the diagram of § 99, $ABCDE$ is a section of the polyhedral angle O .

102. Convex Figures. Either a polyhedral angle, or a polyhedron, is said to be *convex* if no part of the plane of any one of its faces lies within the space inclosed by its faces. As only convex figures will be dealt with in this book, the word *convex* will be understood whenever polyhedral angles or polyhedrons are used.

103. The General Polyhedron. Elementary geometry is concerned mostly with certain classes of polyhedrons, such as prisms, parallelepipeds, and pyramids, all of which will be defined when they are to be used. The general polyhedron is discussed to some extent in § 323.

104. The Prismatic Surface and the Cylindrical Surface. If a moving straight line always remains parallel to its original position, and always intersects a given straight line, it evidently describes, or *generates*, a plane. Why?

In the figures on p. 281, AB represents the original position of the moving line (or *generatrix*), while XY represents the guiding line (or *directrix*).

A straight line that always remains parallel to its original position, and always intersects a line that is in one plane, but is not coplanar with the moving line,

(a) if the guiding line is *broken*, generates a *prismatic surface*; which is evidently composed of a number of por-

PLANE

PRISMATIC SURFACE

CYLINDRICAL SURFACE

tions of planes intersecting in parallel lines, which are the *edges* of the prismatic surface.

(b) if the guiding line is *curved*, generates a *cylindrical surface*, of which no part is plane, for if it were, that part of the guiding line would be straight. The moving line in each of its various positions is called an *element* of the cylindrical surface. For some purposes, it is convenient to let "element" also apply to prismatic surfaces.

A non-coplanar guiding line may give a combination of prismatic and cylindrical surfaces, and need not be considered in an elementary course.

105. Closed Surfaces. If the guiding line is a closed line, the prismatic or cylindrical surface is also *closed*, and the space inclosed is called *prismatic* or *cylindrical space*.

106. Sections. If a plane that is not parallel to the moving line meets a closed prismatic or cylindrical surface,

it cuts all elements and so cuts the surface in a closed line. The figure bounded by this closed line is called a *section* of the prismatic or cylindrical space. If it is perpendicular to an edge or element, it is a *right section*; otherwise it is an *oblique section*. When two sections are made by parallel planes, they are called *parallel sections*. In the figure, $ABCD$ and $EFGH$ are parallel sections of the prismatic space; XY is a section of the cylindrical space.

* 107. *The edges of a prismatic surface, or the elements of a cylindrical surface, between parallel sections are equal.*

* 108. *Parallel sections of a prismatic space are congruent polygons.*

109. **Theorem I.** *Parallel sections of a cylindrical space are congruent.*

Let planes M and N make parallel sections of a cylindrical space. Take points A and B on the perimeter of section M , and let P represent any third point of that perimeter. Draw the elements through A , B , and P to A' , B' , and P' on the perimeter of section N , and draw

AB , AP , BP , $A'B'$, $A'P'$, $B'P'$. Then $\triangle ABP \cong \triangle A'B'P'$ (why?), and if section M is superposed on section N with AB coinciding with $A'B'$, P falls on P' . But P may be any point on the perimeter of section M , so each point of section M falls on a corresponding point of section N , and conversely, and the two sections coincide and are congruent.

110. Prisms and Cylinders. The part of a prismatic space between two parallel sections is called a *prism*. The part of a cylindrical space between two parallel sections is called a *cylinder*. The parallel sections are the *bases*, while the prismatic or cylindrical surface is the *lateral surface* of the figure. The area of the lateral surface is the *lateral area* of the figure. In the case of the prism, the planes forming the lateral surface are *lateral faces*; their intersections are *lateral edges*. A prism or a cylinder is *right* or *oblique* according as its bases are right or oblique sections.

111. Altitude. The *altitude* of a prism or a cylinder is the perpendicular between its bases.

112. Classification of Prisms according to Number of Faces. A prism is *triangular*, *quadrangular*, *pentagonal*, etc., according as its bases have three, four, five, etc., sides.

113. Regular Prisms. A right prism whose bases are regular polygons is a *regular prism*.

114. Circular Cylinders. A cylinder whose bases are circles is a *circular cylinder*; if its bases are also right sections, it is a *right circular cylinder*, or since it might be generated by revolving a rectangle around one side as an axis, it is sometimes called a *cylinder of revolution*. The cylinders considered in elementary solid geometry are almost always circular cylinders. See Appendix, § 321.

115. Axis and Radius of a Circular Cylinder. The line joining the centers of the bases of a circular cylinder is called its *axis*. The radius of the base of a circular cylinder is called the *radius of the cylinder*.

115. Find the locus of points at a fixed distance from a given line.

116. The Cylinder the Limiting Case of a Prism. A prism is said to be *inscribed in*, or *circumscribed about* a cylinder if its bases are respectively inscribed in, or circumscribed about, the bases of the cylinder. The following statement is assumed without proof. Its truth is manifest, but it is difficult to prove.

Theorem II. *If a prism, the base of which is a regular polygon, is inscribed in, or circumscribed about, a given circular cylinder, the lateral area of the prism approaches the lateral area of the cylinder as its limit, and the volume of the prism approaches the volume of the cylinder as its limit, as the number of faces of the prism is increased indefinitely.*

It follows that the circular cylinder is the limiting case of the prism having a regular base, and that theorems true for the one will usually be true for the other. This is why the two solids are here treated together.

117. Theorem III. *A section of a cylinder made by a plane containing an element is a parallelogram.*

116. A section of a prism made by a plane parallel to an edge is a parallelogram.

118. Tangent to a Cylinder. A plane is *tangent to a cylinder* if it meets its surface in one element, and in no point outside that element.

* 119. *If a plane is tangent to a circular cylinder, its intersection with the planes of its bases are tangent to the bases.*

For the intersection has but one point on the circumference.

* 120. *The plane determined by a tangent to a base of a circular cylinder, and the element from the point of contact, is tangent to the cylinder.*

For if it met the cylinder in a point outside the element what would follow?

121. **Theorem IV.** *The lateral area of a prism is the product of an edge and the perimeter of a right section.*

What form would this take for a right prism?

122. **COR.** *The lateral area of a right circular cylinder is the product of an element and the circumference of the base.*

Inscribe a regular prism of lateral area L' , edge e , and perimeter of the base p , in the right circular cylinder of lateral area L , element e , and circumference of the base c . Then $L' = ep$. If the number of faces of the prism is increased without limit, $L \doteq L'$, and $p \doteq c$. Why? But if a variable approaches a limit, its product by a constant will approach the product of its limit by that constant, therefore $ep \doteq ec$. Also, since L' and ep are two variables that are constantly equal, and are approaching the limits L and ec , those limits are equal, that is, $L = ec$. If the cylinder is of radius r and altitude h , its lateral area $= 2\pi rh$.

117. Prove, without using the formula for the lateral area, that the lateral areas of two prisms cut from the same prismatic space and having an equal lateral edge are equivalent.

118. Prove, without using the formula for the lateral area, that the lateral areas of two cylinders cut from the same cylindrical space and having an equal element are equivalent.

119. Find the lateral area of a right prism of altitude 5 in., and having an equilateral triangle of side 4 in. as its base.

120. Find the total area (lateral area and bases) of a regular hexagonal prism of edge (base and lateral) 6 in.

121. Find the lateral area of an oblique prism of edge 10 ft., having as its right section a square of side 2.5 ft.

122. What is the lateral area of a right circular cylinder of element 12 in. and base of radius 3 in.?

123. The pillars of a colonial house are regular hexagonal prisms of base edge 1 ft., and height 30 ft. How much surface must be painted on four of them?

124. How much is the surface of a log 14 in. in diameter and $12\frac{1}{2}$ ft. long diminished by cutting the largest possible square timber out of it?

125. The circular right section of a cylinder has an area of 54 sq. ft., its altitude is 25 ft., and its elements make a 60° angle with the base. Find the lateral area.

123. Theorem V. *Two right prisms, or two right cylinders, are congruent if they have congruent bases and equal altitudes.* Superpose.

124. Equivalence. Two solids are said to be *equivalent* when their surfaces inclose the same amount of space.

126. Two cylinders cut from the same space and having an equal element are equivalent. Show that the solid from the lower base of one to the lower base of the other can slide along the cylindrical space so as to coincide with the solid between the upper bases, etc.

127. An oblique cylinder is equivalent to a right cylinder whose base is its right section, and whose altitude is its element.

125. The Parallelepiped. A prism whose bases are parallelograms is called a *parallelepiped*. It is manifest from this definition that all the faces of a parallelepiped

are parallelograms, and that it is therefore bounded by three pairs of parallel planes, and has three sets of four equal parallel edges. Explain in full.

* **126.** *Any pair of opposite faces of a parallelepiped may be considered to be its bases.* For the parallelepiped fulfills the conditions of the definition of a prism whichever faces are used as bases. Therefore, *the opposite faces of a parallelepiped are congruent.*

127. Right and Rectangular Parallelepipeds. A parallelepiped whose lateral edges are perpendicular to its bases is, by the definition of a right prism, a *right parallelepiped*; otherwise it is *oblique*. If, in addition to being right, it has rectangular bases, it is called a *rectangular parallelepiped*, for all its faces are rectangles. Many shorter names for rectangular parallelepipeds have been suggested but no one is in general use. The term *cuboid* seems as convenient as any.

128. Cube. A rectangular parallelepiped having all its edges equal, and therefore all its faces square, is called a *cube*.

129. Diagonal. The sect joining any vertex of a parallelepiped to the one not in the same face is called a *diagonal* of the parallelepiped.

130. Dimensions. In a rectangular parallelepiped the lengths of the three edges drawn from one vertex are the *dimensions* of the figure.

131. Ratio between Solids. By "the ratio of one solid to another" is meant "the ratio of the amounts of space

occupied by them." This can of course, be worked out only in terms of congruent parts of the solids.

NOTE. At the discretion of the teacher, §§ 132, 133, and 134, may be omitted and § 136 may be assumed as the definition of the volume of a rectangular parallelepiped. The author does not recommend this course, but it has the sanction of some authorities.

132. Theorem VI. *Two rectangular parallelepipeds having two dimensions in common are to each other as their third dimensions.*

For the commensurable case, divide the third dimensions by a common divisor, pass planes through the points of division dividing the given parallelepipeds into congruent parts, and prove the proportion.

The incommensurable case can be proved by limits, but it is not often required in an elementary course.

133. Theorem VII. *Two rectangular parallelepipeds having one dimension in common are to each other as the products of their other dimensions.*

Compare each with a third parallelepiped so constructed as to make this possible.

134. Theorem VIII. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.* Use the method of Theorem VII.

128. Prove Theorem VIII without using Theorem VII, by constructing two auxiliary parallelepipeds with which to compare the given ones.

135. Volume: Cubic Unit. The unit of measure for solid figures is the *cubic unit*, or *unit of volume*, which is a cube whose edge is a linear unit, and whose faces are therefore units of area. Since volume is always expressed in cubic units, the *volume* of any solid is its ratio to a cubic unit, or, in other words, the number of cubic units it contains.

136. Theorem IX. *The volume of a rectangular parallelepiped is the product of its three dimensions.*

Let R be a parallelepiped of dimensions a, b, c , and let U be a unit of volume, of edges equal to the linear unit u . Then compare the two parallelepipeds :

$$\frac{R}{U} = \frac{a}{u} \times \frac{b}{u} \times \frac{c}{u},$$

But $\frac{R}{U}$ is the volume of the parallelepiped, and $\frac{a}{u}, \frac{b}{u},$ and $\frac{c}{u}$ are the three dimensions in length units, therefore,

The volume of a rectangular parallelepiped (in cubic units) equals the product of its three dimensions (in length units). Evidently this volume might also be expressed as the product of its base and its altitude. Why?

129. How many cubic yards of earth must be removed in digging a rectangular ditch 6 ft. deep, 3 ft. wide, and one quarter of a mile long?

130. A log 2 ft. in diameter and 24 ft. long has been sawed in two lengthwise, making two semicircular timbers. If the largest possible square timber is cut out of each of these parts, how many board feet (1 ft. square and 1 in. thick) will they contain?

131. In ex. 130, if the largest possible square log had been cut from the original log, how would the number of board feet in it compare with the number in the two timbers?

137. Theorem X. *If two solids contained between the same two parallel planes are such that their sections by any third plane parallel to these two planes are equivalent, the two solids have the same volume.*

This is "Cavalieri's Theorem," and such solids as those described are called *Cavalieri bodies*. The theorem depends on limits for its proof, but the proof is too difficult at this point. See Appendix, § 319.

To prove two solids equivalent in volume by this theo-

rem, it is necessary (1) to show that the solids can be placed between the same parallel planes; (2) to pass any third parallel plane through the solids, and to prove that the sections obtained are equivalent.

138. Theorem XI. *Prisms cut from the same prismatic space and having equal lateral edges are equivalent in volume.*

For the part of the prismatic space between their upper bases can coincide with the part between their lower bases; and after taking each of these parts from the whole figure formed by the prisms, equivalent prisms remain. Draw the prisms so that their bases do not intersect, as this would unnecessarily complicate the figure. They may be entirely outside each other, or may have some common space. Note the similarity between this proof and one method of proving that parallelograms having equal bases and lying between parallels are equivalent.

139. Theorem XII. *A parallelepiped is equivalent to a rectangular parallelepiped of equivalent base and equal altitude.*

For, given the oblique parallelepiped R , with the base $ABCD$, and the altitude h , pass two planes M and N , whose distance apart, XY , equals AB , through the prismatic space of which R is a part, so as to be perpendicular to AB .

Then these planes cut from the prismatic space a new parallelepiped S equivalent to R , since $XY = AB$, and having its rectangular base $XYZW$ equivalent to $ABCD$ (why?), and the same altitude h . In S , the planes M and N are perpendicular to $XYZW$, but the other lateral faces may or may not be perpendicular to the base.

The same operation applied to S by extending YZ and its parallels to form the edges of a prismatic space, and passing planes perpendicular to YZ and the distance YZ apart, would form a third parallelepiped equivalent to S and therefore to R , and having a base equivalent to the bases of those figures, and the same altitude h . But this parallelepiped would, by construction, be rectangular, for its base would be a rectangle, and the lateral planes would all have been constructed perpendicular to the base. Therefore it is possible to transform any parallelepiped into an equivalent rectangular parallelepiped with an equivalent base and the same altitude.

But one of the transformations is shown in this figure, because the attempt to repeat the operation in the same figure makes an unnecessarily complicated diagram.

Alternative method: Use Cavalieri's Theorem.

140. COR. *The volume of any parallelepiped is the product of its base and its altitude.*

141. Theorem XIII. *A plane through two opposite edges of a parallelepiped divides it into equivalent triangular prisms.*

If the parallelepiped is right, the prisms are congruent. If not, it is equivalent to a right parallelepiped cut from the same space, and the parts into which the plane cuts it are equivalent to the halves of that right parallelepiped.

142. COR. 1. *The volume of a triangular prism is the product of its base and its altitude.*

143. COR. 2. *The volume of any prism is the product of its base and its altitude.*

144. COR. 3. *The volume of a circular cylinder is the product of its base and its altitude.* For it is the limit approached by the inscribed prism of regular base when the number of faces is increased indefinitely. (See § 116.) The volume of a cylinder of radius r and altitude h is $\pi r^2 h$.

132. Which would be lighter and how much, circular pillars 23 in. in diameter and 12 ft. high, or octagonal pillars 24 in. in diameter and 12 ft. high, if they were made of cement weighing 150 pounds to the cubic foot?

133. What per cent of the lumber is cut off in cutting a square timber out of a log of radius r ? in cutting out a hexagonal timber?

134. The diagonals of a parallelepiped are concurrent in the midpoint of each.

135. Any section of a parallelepiped through two pairs of parallel planes is a parallelogram.

136. If the diagonals of the faces of a parallelepiped are drawn, the lines joining the intersection points of the diagonals in opposite faces are concurrent in the midpoint of each.

137. The diagonal of a cube equals the square root of three times its edge.

138. The sum of the squares on the diagonals of a parallelepiped equals the sum of the squares on its edges.

139. The square on a diagonal of a rectangular parallelepiped equals the sum of the squares on its three edges.

140. Find the volume of a circular cylinder of radius 5 cm., whose axis is 1.5 m. long, and makes an angle of 45° with its bases.

141. An irrigation ditch is 4 ft. wide, 3 ft. deep for its whole width, with a semicylindrical bottom below this 3 ft. If it is filled to within 6 in. of the top, how much water does it contain to the mile?

SECTION II. THE PYRAMID AND THE CONE

145. The Pyramidal Surface and the Conical Surface.

If a moving straight line always contains a fixed point, and always intersects a given straight line, it generates a portion of a plane. Why?

In these figures, AB represents the original position of the moving line (or *generatrix*), P being the fixed point. XY represents the guiding line (or *directrix*).

A moving straight line that always contains a fixed point, and always intersects a line that lies entirely in one plane, but is not coplanar with the moving line:

(a) if the guiding line is *broken*, generates a *pyramidal surface*, which is evidently composed of portions of a number of planes having one common point, with each successive pair intersecting in lines that are the *edges* of the surface.

(b) if the guiding line is *curved*, generates a *conical surface*, of which no part is plane, for if it were, that part of the guiding line would be straight. (Why?) The mov-

ing line in each of its various positions is called an *element* of the conical surface.

Non-coplanar guiding lines may give combinations of pyramidal and conical surfaces, and need not be considered in an elementary course.

146. Vertex. The fixed point through which the moving line passes is called the *vertex*.

147. Closed Surfaces. If the guiding line is a closed line, the pyramidal, or conical, surface also is closed, and the space inclosed is called *pyramidal*, or *conical*, *space* respectively.

148. Sections. If a plane cuts all the edges of a closed pyramidal surface, or all the elements of a closed conical surface, but not at the vertex, it cuts the surface in a closed line, and the figure bounded by this closed line is called a *section* of the pyramidal or the conical space.

When two sections are made by parallel planes, they are called *parallel sections*.

S_1 and S_2 are parallel sections of the closed pyramidal space P , and of the closed conical space C . In each case V is the vertex. In the pyramidal space the section must be a polygon. Why?

***149.** *The edges from the vertex of a pyramidal surface, or the elements from the vertex of a conical surface, are cut proportionally by parallel sections.*

150. Theorem XIV. *Parallel sections of a pyramidal space are similar polygons, and their areas are proportional to the squares of their distances from the vertex.*

142. Which are the corresponding points on the perimeters of parallel sections? If a line is drawn from the vertex of a conical space through two parallel sections, show that the sects from the points thus obtained to any two corresponding points on the perimeters of the sections will be proportional to the distances of those sections from the vertex.

151. Pyramids and Cones. That part of a pyramidal space between the vertex and a section is called a *pyramid*. That part of a conical space between the vertex and a section is called a *cone*. The section is called the *base*, and the pyramidal or conical surface is called the *lateral surface* of the figure, while the area of the lateral surface is called the *lateral area*. In the case of the pyramid, the portions of planes forming the lateral surface are called *lateral faces*, and their intersections are called *lateral edges*.

152. Truncated Figures. That part of a prismatic or cylindrical space between two non-parallel sections that do not intersect on the surface is called a *truncated prism* or *cylinder*.

That part of a pyramidal or conical space between two sections on the same side of the vertex that do not intersect on the surface is called a *truncated pyramid* or *cone*.

153. Frustums. A truncated pyramid or cone between parallel sections is called a *frustum* of the pyramid or cone.

154. Lateral Faces and Edges. The definition of lateral surface, lateral faces, and lateral edges given in § 151 holds also for frustums and truncated figures. The parallel sections in a frustum are its *bases*, but only one of the sections is in general considered the *base* of the truncated figure.

155. Altitude. The *altitude* of a pyramid or a cone is the perpendicular from the vertex to the base. In a frus-

tum, it is the perpendicular between the bases. A truncated figure cannot be said to have an altitude; a triangular truncated prism, however, is sometimes said to have three altitudes, namely, the perpendiculars from the three vertices of one section to the other section.

156. Classification of Pyramids according to Number of Faces. A pyramid, like a prism, is *triangular*, *quadrangular*, *pentagonal*, etc., according to the number of sides of its base.

157. Regular Pyramids. A pyramid whose base is a regular polygon, and whose altitude meets the base at its center, is called a *regular* pyramid. The altitude of such a pyramid is called its *axis*. The lateral faces of a regular pyramid are congruent. (Why?). The altitude of one of the lateral faces is called the *slant height* of the pyramid.

158. Circular Cones. A cone whose base is a circle is a *circular cone*. The line joining the vertex of a circular cone to the center of its base is called the *axis* of the cone, and if the axis is perpendicular to the base, the cone is a *right circular cone*. The radius of the base is called the *radius of the cone*. The cones considered in elementary solid geometry are usually circular cones. See Appendix, § 321.

The right circular cone, since it might be generated by revolving a right triangle about one leg as an axis, is sometimes called a *cone of revolution*. An element of a right circular cone is sometimes called its *slant height*.

143. Find the locus of a point such that its distances from a given plane and a line perpendicular to that plane shall have a given fixed ratio.

159. The Cone the Limiting Case of the Pyramid. A pyramid is said to be *inscribed in* or *circumscribed about* a cone if the pyramid and cone have the same vertex and the base of the pyramid is respectively inscribed in or circumscribed about the base of the cone. The following theorem is assumed without proof:

Theorem XV. *If a pyramid, the base of which is a regular polygon, is inscribed in or circumscribed about a given circular cone, the lateral area of the pyramid approaches the lateral area of the cone as its limit, and the volume of the pyramid approaches the volume of the cone as its limit, as the number of sides of the base of the pyramid is increased indefinitely.*

The cone is then a limiting case of the pyramid; therefore the two figures will be treated together.

160. Theorem XVI. *A section of a cone made by a plane containing an element is a triangle.*

144. In Theorem XVI, would it be sufficient to say "by a plane through the vertex"?

145. A section of a pyramid made by a plane through the vertex is a triangle.

161. Theorem XVII. *A section of a circular cone made by a plane parallel to the base is a circle, the center of which is the point where the axis meets it.*

Show that the sects from this point to the perimeter of the sections are in proportion to the radii of the base.

162. Tangent to a Cone. A plane is *tangent to a cone* if it meets its surface in one element, and in no other point.

***163.** *If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.*

* 164. *The plane determined by a tangent to the base of a circular cone and the element from the point of contact, is tangent to the cone.*

165. **Theorem XVIII.** *The lateral area of a regular pyramid is equal to one half the product of the slant height and the perimeter of the base.*

166. **COR.** *The lateral area of a right circular cone is equal to one half the product of the slant height (or element) and the circumference of the base. If the cone is of radius r and slant height s , the lateral area $= \pi rs$.*

167. **Theorem XIX.** *The lateral area of a frustum of a regular pyramid is equal to one half the product of the slant height and the sum of the perimeters of the bases. What kind of plane figures are its faces?*

168. **COR. 1.** *The lateral area of a frustum of a right circular cone is equal to one half the product of the slant height and the sum of the circumferences of the bases. If the radii of the bases are r_1 and r_2 , and the slant height is s , the lateral area is $\pi s(r_1 + r_2)$.*

169. **Midsections.** When a solid has parallel bases, the section parallel to the bases and half way between them is called the *midsection*.

170. **COR. 2.** *The lateral area of a frustum of a right circular cone is equal to the product of the slant height and the circumference of its midsection.*

If the radius of the midsection is r_m , lateral area $= 2 \pi r_m s$.

171. **COR. 3.** *The lateral area of a frustum of a right circular cone is equal to the product of the altitude and the circumference of a circle whose radius is the perpen-*

dicular erected at the midpoint of an element, and terminated by the axis. For if the perpendicular is a and the altitude h , by similar triangles $\frac{s}{h} = \frac{a}{r_m}$, and ah can be substituted for sr_m in the formula of § 170, giving $2\pi ah$.

146. Find the surface of a hexagonal tower of base edge 10 ft. and altitude 20 ft. surmounted by a pyramid of height 15 ft.

147. What area does a line generate if it is revolved around a coplanar axis, its projection upon the axis being 10 in. and the length of its perpendicular bisector when extended to the axis being 6 in.?

172. Prisms Inscribed in, and Circumscribed about, a Pyramid. If the altitude of a triangular pyramid is divided into equal parts by planes passed parallel to the base,

1. The prisms within the pyramid having the sections formed by the parallel planes as bases, and each having one of the divisions of a certain edge of the pyramid as one of its lateral edges, are called a *set of inscribed prisms*. In Fig. 1, R , S , and T are inscribed prisms, the common edge being VC .

FIG. 1.

FIG. 2.

2. The prisms having the base of the pyramid and the parallel sections as bases, and each having one of the divisions of a certain edge of the pyramid as one of its lateral edges, are called a *set of circumscribed prisms*. In Fig. 2, X , Y , Z , and W are circumscribed prisms, the common edge being VC .

When a set of circumscribed prisms and a set of inscribed prisms are formed in a pyramid by the use of the same lateral edge and the same parallel planes, they are spoken of as *corresponding sets of inscribed and circumscribed prisms*.

***173.** *The volume of a triangular pyramid is greater than that of any set of inscribed prisms, and less than that of any set of circumscribed prisms.*

***174.** *In any triangular pyramid, the difference in volume between a set of circumscribed prisms and the corresponding set of inscribed prisms is the circumscribed prism on the base of the pyramid.*

As, in the figures, $R = Y$, $S = Z$, $T = W$, therefore

$$(X + Y + Z + W) - (R + S + T) = X.$$

***175.** *If the number of equal parts into which the altitude of a triangular pyramid is divided by planes parallel to the base is increased indefinitely, the volumes of the sets of inscribed and circumscribed prisms formed with the sections as bases will approach the volume of the pyramid as a limit.*

For, since their difference is a prism on a constant base (ABC) whose altitude can be made indefinitely small, the difference between the two sets, and therefore between either and the pyramid, can be made to approach zero.

176. Theorem XX. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent in volume.*

Inscribe sets of prisms, using the same number of divisions of the altitudes. What is known of the volumes

of correspondingly placed prisms in the two pyramids? What is true of the volumes of the sets of prisms? Increase the number of divisions of the altitude. Then what is true of the sets of prisms? Therefore what is true of the pyramids? Why?

Alternative Method. The two pyramids are Cavalieri bodies.

177. Theorem XXI. *A triangular prism can be divided into three equivalent triangular pyramids.*

Pass two planes through vertices of the prism so as to divide it into three pyramids. See whether they are equivalent by examining their bases and altitudes. It is possible to prove any one equivalent to any other. When no easier method seems possible, choose the bases of the two to be examined so that they are in the same plane.

178. COR. 1. *The volume of a triangular pyramid equals one third the product of its base and its altitude.*

179. COR. 2. *The volume of any pyramid equals one third the product of its base and its altitude.*

180. COR. 3. *The volume of a circular cone equals one third the product of its base and its altitude.* $V = \frac{1}{3} \pi r^2 h.$

148. Find the volume of a right circular cone of radius 8 in. and altitude 15 in., and of the regular square pyramid circumscribed about it.

149. If square pyramids whose lateral faces are equilateral triangles are placed on all the faces of a cube, find the volume of the resulting figure.

150. If a rectangular parallelepiped of base 6 in. by 8 in., and altitude 14 in. is hollowed out along the diagonal planes from its bases to the intersection of its diagonals, find the volume of the remaining figure.

151. If the corners are cut off a cube by passing planes through the one-third points on the three edges from each vertex, what part of its volume is left?

152. Find the volume of the solid formed by passing planes through the midpoints of the three edges of each trihedral angle of a cube.

181. Prismatoids. A polyhedron all of whose vertices lie in two parallel planes is called a *prismatoid*. Its bases are obviously polygons in parallel planes, and its lateral faces are quadrilaterals or triangles.

182 The Prismatoid Formula and its Extension to Curved Surfaces.

The volume of any prismatoid is expressed by the formula:

$$V = \frac{h}{6}(b_1 + b_2 + 4m),$$

where h is the altitude, b_1 and b_2 are the bases, and m is the midsection.

Extension of the Formula. The formula applies also to solids having curved surfaces, if (1) their bases are circles in parallel planes; (2) their generating elements are straight lines or arcs of circles.

This extends the formula to cylinders and cones, and to certain parts of spheres that have not yet been studied; it also applies to other figures with which elementary geometry is not concerned.

The proof of the Prismatoid Formula is not given, but for the case of polyhedrons it will be found in the Appendix, § 320. It will not be used to any extent to prove other formulas, but will be shown to agree with the formulas found for the various figures by other methods. It serves, then, as a summary for almost all the volume formulas of

solid geometry, and might well be used for all the volumes to which it applies were it not for the fact that it is often harder to apply than the particular formula for the figure in question.

For example, in the cone, the midsection is the base divided by 4 and one base (considering the cone a prismatoid) is zero, so the prismatoid formula would be

$$\frac{h}{6} \left(b + 4 \times \frac{b}{4} \right) \text{ or } \frac{hb}{3},$$

which agrees with the regular formula.

183. Theorem XXII. *The volume of the frustum of a pyramid equals one third the product of its altitude and the sum of its two bases and the mean proportional between them.*

$$V = \frac{h}{3} (b_1 + b_2 + \sqrt{b_1 b_2}),$$

h being the altitude, and b_1 and b_2 the lower and upper bases, the one farther from the vertex being considered, for convenience, the lower base.

FIRST PROOF. Continue the lateral edges through the vertices of the frustum to the vertex of the pyramid, and consider the frustum as the difference of two pyramids. The altitude of the smaller pyramid can be obtained from the proportion $\frac{b_1}{b_2} = \frac{(h+x)^2}{x^2}$, where x is the unknown altitude. Solve this for x by taking the square root and applying division, finding $x = \frac{h \sqrt{b_2}}{\sqrt{b_1} - \sqrt{b_2}}$. Express the volume of the frustum as the difference of two pyramids in terms of h , x , b_2 , b_1 , and substitute this value of x .

SECOND PROOF. Divide a triangular frustum into three triangular pyramids, one of which has the lower base of the frustum and the same altitude, a second of which has the upper base of the frustum and the same altitude, and prove that the third is the mean proportional between these two. In order to do this take the ratio of the third pyramid to each of the others, and reduce these ratios to simpler form. Use the same method as in Theorem XXI, but the bases of the pyramids, instead of being equal, will have equal altitudes only, and so, in turn, will be proportional to their bases. This proof must be extended to any frustum.

THIRD PROOF. Use the Prismatoid Formula, expressing the midsection in terms of the bases, to which it is similar. This is done by taking the ratio of each of the bases to the midsection and adding the square roots of the equations obtained.

It is recommended that all of these proofs be worked out, for they are all valuable applications of the various methods used.

184. COR. *The volume of a frustum of a circular cone is equal to one third the product of its altitude and the sum of its two bases and the mean proportional between them.*

$$V = \frac{\pi h}{3} (r_1^2 + r_1 r_2 + r_2^2),$$

where h is the altitude, and r_1 and r_2 are the radii of the bases.

153. A monument is in the form of a pillar of granite 1 ft. in diameter and 12 ft. high, surmounted by the frustum of a cone 1 ft. high with an upper base of radius 10 in., and a cone of height 6 in. How much does the stone weigh at 175 lb. to the cubic foot?

185. SUMMARY OF PROPOSITIONS

I. THE STRAIGHT LINE :

(1) *Sects equal* :

The edges of a prism are equal (107).

The elements of a cylinder are equal (107).

(2) *Sects proportional* :

The edges of a pyramidal space cut by parallel planes are proportional (149).

II. TANGENT PLANES AND SECTIONS :

(1) *Tangent planes* :

The plane determined by an element and a tangent to the base of a cylinder (or a cone) is tangent to the cylinder (or the cone), and conversely (119, 120, 163, 164).

(2) *Sections* :

(a) of a prismatic space :

Parallel sections are congruent polygons (108).

Any pair of opposite faces of a parallelepiped may be considered its bases (126); its opposite faces are congruent (126).

(b) of a cylindrical space :

Parallel sections are congruent (109).

A section through an element of a cylinder is a parallelogram (117).

(c) of a pyramidal space :

Parallel sections are similar polygons (150); their areas are proportional to the squares of their distances from the vertex (150).

(d) of a cone :

A section parallel to the base of a circular cone is a circle (161).

A section through an element is a triangle (160).

III. LIMITING CASES :

If the number of faces of a prism (or a pyramid) of regular base, inscribed in or circumscribed about a circular cylinder (or a cone), is increased indefinitely, the lateral area and the volume

of the prism (or the pyramid) approaches as its limit the lateral area and the volume of the cylinder (or the cone) (116, 159). If the number of equal parts into which the altitude of a triangular pyramid is divided by planes parallel to the base is increased indefinitely, the volumes of the sets of inscribed and circumscribed prisms formed with the sections as bases approach the volume of the pyramid as a limit (175).

IV. LATERAL AREA:

- (1) of a prism = edge $\times$ perimeter of a right section (121);
- (2) of a right circular cylinder = altitude $\times$ perimeter of the base = $2\pi rh$, where h is the altitude, r is the radius (122);
- (3) of a regular pyramid = $\frac{1}{2}$ slant height $\times$ perimeter of the base (165);
- (4) of a right circular cone = $\frac{1}{2}$ slant height $\times$ perimeter of the base = πrs , where r is the radius, s is the slant height (166);
- (5) of a frustum of a regular pyramid = $\frac{1}{2}$ slant height $\times$ the sum of the perimeters of the bases (167);
- (6) of a frustum of a right circular cone = $\frac{1}{2}$ slant height $\times$ the sum of the perimeters of the bases = $\pi s(r_1 + r_2)$, where r_1 and r_2 are the radii, = $2\pi sr_m$, where r_m is the radius of the midsection, = $2\pi ah$, where a = the perpendicular bisector of an element extended to the axis (168, 170, 171).

V. VOLUMES:

- (1) *Congruent*:

Right prisms or cylinders having congruent bases and equal altitudes are congruent (123).

- (2) *Equivalent*:

Two Cavalieri bodies are equivalent (137).

Prisms cut from the same prismatic space and having equal lateral edges are equivalent (138).

An oblique parallelepiped is equivalent to a rectangular parallelepiped having an equivalent base and an equal altitude (139).

The parts of a parallelepiped made by passing a plane through two opposite edges are equivalent (141).

Triangular pyramids of equivalent bases and equal altitudes are equivalent (176).

A triangular prism is equivalent to three equivalent pyramids (177).

(3) *Proportional:*

Rectangular parallelepipeds having two dimensions equal are to each other as the third dimension (132).

Rectangular parallelepipeds having one dimension equal are to each other as the products of the other two dimensions (133).

Rectangular parallelepipeds having no dimensions equal are to each other as the products of all three dimensions (134).

(4) *The Pyramid and its inscribed and circumscribed prisms:*

A triangular pyramid is less than the circumscribed, and greater than the inscribed prisms (173).

The difference between the sets of circumscribed and inscribed prisms is the circumscribed prism on the base of the pyramid (174).

(5) *Formulas for volume:*

(a) of a rectangular parallelepiped = the product of its dimensions = base $\times$ altitude (136);

(b) of any parallelepiped = base $\times$ altitude (140);

(c) of a prism = base $\times$ altitude (142, 143);

(d) of a circular cylinder = base $\times$ altitude (144);

(e) of a triangular pyramid = $\frac{1}{3}$ base $\times$ altitude (178);

(f) of any pyramid = $\frac{1}{3}$ base $\times$ altitude (179);

(g) of a circular cone = $\frac{1}{3}$ base $\times$ altitude (180);

(h) of a prismatoid = $\frac{1}{6}$ altitude $\times$ the sum of the bases and four times the midsection (182);

(i) of a frustum of a pyramid = $\frac{1}{3}$ altitude $\times$ the sum of the bases and the mean proportional between them (183);

(j) of a frustum of a circular cone = $\frac{\pi}{3} \times$ altitude $\times$ the sum of the squares of the radii and their product, which

$$= \frac{\pi h}{3} (r_1^2 + r_2^2 + r_1 r_2) \text{ where } h \text{ is the altitude, and } r_1$$

and r_2 are the radii of the bases (184).

Explain why the circular cylinder and cone can be assumed to be the limits of the inscribed or the circumscribed prism and pyramid with regular bases. How is this fact used? What changes are made in the formulas when they are applied to the circular figures? What application of similar figures occurs in this book? If the altitude of a pyramid is divided into ten equal parts by planes passed parallel to the base, what ratio has the area of each of the sections to the area of the base? State the theorem used, and explain briefly why it is true. State formulas for the lateral areas of prism, right prism, parallelepiped, right parallelepiped, rectangular parallelepiped, circular cylinder, right circular cylinder, right circular cone, regular pyramid, frustum of a regular pyramid, frustum of a right circular cone. Why must it be a *regular* pyramid? why a *right* circular cone? Define a plane tangent to a cylinder or a cone. How is it determined? Explain through what succession of propositions the volume of a rectangular parallelepiped is found. Follow the steps from the volume of a rectangular parallelepiped to that of any pyramid. Does the pyramid need to be regular for this formula to hold? Does the cone need to be circular for its volume formula to be true? Explain in full. State Cavalieri's Theorem. What facts need to be established about two figures in order that this theorem can be used? When does the Prismatoid Formula hold? State the volume formulas for all the solids studied in this book. What is known of parallel sections of a prism? of a cylinder? of a pyramid? of a cone? of a section through an edge of a prism? through an edge of a pyramid? through an element of a cylinder? through an element of a cone? through two opposite edges of a parallelepiped? cutting two pairs of its opposite faces? through three concurrent edges? What sections are congruent?

GENERAL EXERCISES

154. If three planes are perpendicular to each other, the square of the distance from any point to their common vertex equals the sum of the squares of the distances from that point to the three planes.

155. If at the top of a chimney whose flue is 8 in. by 10 in. a round pipe is to be placed, how large should it be to give the same opening? If two equal round chimneys are to be placed side by side, how large should each be to give the same total opening?

156. What kind of surface is generated by opening a door?

157. If in the base of a right circular cylinder of radius r and altitude h , two diameters are drawn perpendicular to each other, and four planes perpendicular to the base are passed through the ends of each two consecutive radii formed by these diameters, what part of the volume is inclosed by these planes?

158. If a right section of a certain hexagonal pillar of diameter 20 ft. is taken, it will show that each surface is curved toward the center along a 90° arc, all the arcs being equal. The pillar is 100 ft. high, the last 20 ft. being pyramidal, with the base a right section of the lower part of the pillar. If this pillar stands on a frustum of a regular pyramid, which has edges 30 ft. and 20 ft., and the altitude 15 ft., how many cubic yards of material were used in the entire construction?

159. In some hospitals, floors are joined to the walls by a curved surface instead of by an edge. If the concrete is shaped by the use of a trowel in the form of the surface of a cylinder of radius 2 in., used with its convex surface toward the edge, find how much concrete to the linear foot would be needed to change an edge into the curved surface.

160. If concrete blocks are made in the form of a rectangular parallelepiped 6 in. $\times$ 8 in. $\times$ 14 in., and they are hollowed out, parallel to the edges of length 8 in., in the form of a parallelepiped with cylindrical ends, the entire length of the hollow being 10 in., what part of a wall built of these blocks will be air space?

161. Which holds more, a suit case 14 in. $\times$ 2 ft. $\times$ 6 in., or one 13 in. $\times$ 23 in. $\times$ 7 in.?

162. How much air space is gained in a house 20 ft. wide, with stories $8\frac{1}{2}$ ft. and 8 ft. high respectively, by adding two semicircular bay windows across the front, allowing 1 ft. for the thickness of the middle partition, and six inches for each side wall?

163. The sign on a corner store extends 10 ft. along one street, with a quarter circle at the corner, and a 17-ft. extension on the other street. If the sign is 2 ft. wide and its area is 63.4248 sq. ft., what is the radius of the corner?

164. Given a cone of radius r and height h , find what fraction of its volume is lost by cutting out a pyramid with its base a regular

polygon inscribed in the base of the cone, if the polygon has (1) 3 sides, (2) 4 sides, (3) 6 sides, (4) 8 sides.

165. A shed is built 9 ft. high in the front and 7 ft. high in the back, with a floor 12 ft. across and 10 ft. from front to back. What is the air space, the amount of lumber in the sides, and the number of square yards of roofing needed to cover it, if the roof extends 1 ft. all around, and 10 % is allowed for laps and waste?

166. If the diagonal of a cube is $4\sqrt{3}$ in., find its volume.

167. In a right circular cone, of lateral area l and base b , find the altitude.

168. If the lateral area of a right circular cone is equivalent to the area of the base, express the volume in terms of the radius.

169. It has been found that in irrigating by concrete ditches whose sides are perpendicular to the bottom, there is the least possible friction for a given volume of water when the depth of the ditch is half its width. As the friction depends on the amount of bottom and side wall, such a ditch is also most economical in materials. Find the most economical rectangular ditch to carry 56,320 cu. yd. of water to the mile, and show that one carrying the same amount of water, but having either higher or lower side walls, would be less economical. (Do this by finding the total area for each of three cases.)

170. If a ditch is to have its side walls at a 45° slope, it is most economical when the width at the bottom is approximately four fifths of the depth. Find the contents of such a ditch 5 ft. deep and a hundred yards long.

171. It has been found that the most economical cylindrical can to use for canning fruit or vegetables is one whose altitude equals its diameter. Find the volume of such a can 3 in. in height, and show that one of the same volume, but of slightly greater height would require more tin to inclose it.

172. If a cylindrical shell of external radius 6 in. and thickness $\frac{1}{4}$ in., has one conical end, the slant height of the cone being 12 in. and the total length of the cylindrical part of the shell being $24\frac{1}{4}$ in., find the weight of the metal used, at 450 lb. to the cubic foot. (Note that the difference in the altitude of the inside and outside cones is not $\frac{1}{4}$ in.)

173. A rectangular building stone 6 in. by 12 in. by 3 ft. is dropped into a cistern of radius 3 ft. How much does it raise the water?

174. A body of irregular shape was placed in a cylindrical vessel of diameter 6 in. partly filled with water. What was its volume if it raised the water $1\frac{1}{2}$ in.?

175. What would be the smallest amount of tin that would make a cylindrical can to contain one quart, counting $57\frac{3}{4}$ cu. in. to the quart, and allowing 5 % waste for cutting and lapping?

176. The greatest right circular cylinder that can be inscribed in a right circular cone is one whose altitude is one third that of the cone. Show the ratio of its volume to that of the cone, and the ratio of its area to that of the cone.

177. How much water will a pail hold if it is 16 in. deep, a foot in diameter at the bottom, and 14 in. in diameter at the top?

178. Find the volume of a polyhedron having one base a rectangle 6 ft. by 4 ft., the other a square of edge 2 ft., with each side parallel to the corresponding side of the rectangle, and an altitude of 8 ft.

179. A hole is dug in the form of the frustum of a regular square pyramid of upper edge 4 ft., lower edge 2 ft., depth $4\frac{1}{2}$ ft. Find the amount of earth removed. If the rain fills it two thirds of the way up, what part of its volume is still above water?

180. How much tin is required to make a funnel in the form of the frustum of a cone 5 in. high, with radii 3 in. and $\frac{1}{2}$ in., with a cylinder 2 in. long on the smaller end, allowing 10 % for waste?

181. What is the cubic content of a concrete support in the form of a regular frustum with square bases of edges 2 ft. and 18 in., and height 18 in.?

182. A street car has side walls 6 ft. high, above each of which is a quarter circle of radius $1\frac{1}{2}$ ft., the center being inside of, and on the level with, the top of the side wall, and at the end of this arc, a second side wall of height 15 in. If the total width of the car is 8 ft., and its length is 50 ft., how many cubic feet of air are there to a passenger when one hundred and ten passengers are in the car?

183. Measure the room in which your class recites, and calculate the number of cubic feet of air space per person.

184. A suburban railroad, in changing the position of its tracks to avoid grade crossings, inclosed its right of way by concrete walls

12 ft. high, 1 ft. wide at the top, and 3 ft. wide at the bottom, one face being perpendicular to the plane of the tracks. If 12 mi. of track were inclosed, how many cubic yards of concrete were built?

185. How many cubic yards of concrete would be needed to taper off the wall described in Ex. 184 at a 45° angle at the level?

The Wedge. A solid whose base is a rectangle, and two of whose four lateral faces meet in an edge parallel to the base, is called a *wedge*. The edge parallel to the base is called the *edge* of the wedge.

186. A wedge has a base 2 in. by 6 in., the altitude 12 in., and an edge 4 in. Find its volume.

187. Find the volume of a right circular cone whose volume equals its lateral surface times one sixth the radius of the base, which is $\sqrt{3}$ in. long.

188. A cube is hollowed out from one face, in the form of a square pyramid whose lateral faces are equilateral triangles. What part of the cube is left? What is the total area of the figure if the edge is 10 in.?

189. Find the amount of space between the surface of a regular hexagonal pyramid of base edge 1 ft., lateral edge 2 ft., and that of the inscribed cone.

190. If the total area of a cylinder of radius r is a , and its volume is br , express r in terms of a and b .

191. A subway stair has a platform 6 ft. by 9 ft., approached on opposite sides by two flights of stairs of 10 steps each, each flight being 6 ft. wide, and each step having a rise of 7 in., and a tread of 9 in. If the concrete is 2 in. thick, how much air space is there in the closet beneath the platform and the stairs?

192. A steam shovel has a scoop 4 ft. deep, 3 ft. wide, and $4\frac{1}{2}$ ft. long, the cross section being a square with a semicircle on one end. How many times would the scoop have to be filled to dig a cellar 30 ft. wide, 45 ft. long, and 6 ft. deep, with two bow windows, one semi-circular, of diameter 10 ft., the other a three-quarter circle of radius 6 ft., around one corner of the house as a center, if, on the average, the scoop is two thirds full each time?

193. How many cubic feet of masonry are there in a round chimney 50 ft. tall, 8 ft. in external, and 5 ft. in internal, diameter at the bottom, and 5 ft. in external, and 4 ft. in internal, diameter at the top?

194. How much cloth is there in a skirt 24 in. around the waist, 6 ft. around the bottom, and 42 in. long, allowing 10 % for seams and fitting?

195. To cut a cube by a plane so that the section shall be a regular hexagon.

196. A measure is in the form of a right circular frustum of radii 2 in. and 3 in. If it holds two quarts, find its height. If it is graduated to pints, at what points on an element should the graduations be made?

197. What would be the most economical size for a factory fire tank to hold 2000 gal. of water?

198. Pencils $\frac{5}{16}$ in. in diameter have their halves cut from wood $\frac{5}{32}$ in. thick. If the lead is $\frac{1}{8}$ in. in diameter, what per cent of the wood is wasted?

199. How much earth is removed in digging 12 post holes 18 in. in diameter and 5 ft. deep?

200. The midpoints of two pairs of opposite edges of a tetrahedron are coplanar.

201. A section of a tetrahedron by a plane parallel to two opposite edges is a parallelogram.

202. The lines joining the vertices of a tetrahedron to the centroids of the opposite faces meet in the three-fourths point of each.

203. To pass a plane through a given point tangent to a given circular cone.

204. To pass a plane through a given point tangent to a given circular cylinder.

205. If the number of square units in the area of a cube equals the number of cubic units in its volume, find its area.

206. A watering trough was built by nailing two boards together at right angles, one against the edge of the other, and closing the ends. If the boards were 1 in. thick, and the one nailed against the edge of the other was the wider by 1 in., find the amount of water such a trough 1 ft. 5 in. wide across the open top and 8 ft. long would hold.

207. If the radius of one cylinder equals the altitude of a second and *vice versa*, what is the ratio of their volumes?

BOOK VIII. POLYHEDRAL ANGLES AND THE SPHERE

SECTION I. DEFINITIONS; SECANTS AND TANGENTS

187. The Sphere. The geometric solid bounded by a surface all points of which are equidistant from a point within, called the *center*, is a *sphere*. The bounding surface is called a *spherical surface*.

188. Generation of a Sphere. A sphere can be generated by the rotation of a semicircle about its diameter as an axis, for all points on the surface of the solid thus generated will be at a distance from the center of the semicircle equal to a radius.

189. Radii and Diameters. A line from the center of a sphere to its surface is a *radius*, while a line through the center, joining two points on the surface, is a *diameter*.

*** 190.** *All radii of a sphere are equal. All diameters of a sphere are equal.*

*** 191.** *Spheres having equal radii are congruent and conversely.*

For, if superposed with centers coinciding (that is, placed so that they are *concentric*), no point on one surface could be off the other surface. (Why?) Note also that equivalent spheres must have equal radii, for if not, the one having the longer radius could contain the other.

From this it follows that only the word *equal* need be used for spheres, for with them there is no distinction between equivalence and congruence. Compare with the circle.

*** 192.** *The locus of points at a given distance from a given point is the surface of a sphere having the given point as its center, and the given distance as its radius.*

208. Find the locus of the vertex of the right angle of a right triangle, having a given fixed hypotenuse.

193. Relative Positions of a Point and a Sphere. A point is *within a sphere* if its distance from the center is less than the length of the radius; *on the surface*, if its distance from the center is equal to the length of the radius; *outside*, if its distance from the center is greater than the length of the radius. This follows from the definition of a sphere, and the ordinary meanings of "within" and "outside."

*** 194.** *One, and but one, spherical surface can be drawn through four non-coplanar points.* For there is one and but one point equidistant from the four points. See § 91.

195. Inscribed Polyhedrons. A sphere is said to be *circumscribed* about a polyhedron if its surface contains the vertices of the polyhedron; the polyhedron is said to be *inscribed* in the sphere. § 194 might have been worded: *One and but one sphere can be circumscribed about a given tetrahedron.*

196. Relative Positions of a Line and a Sphere. A line can lie wholly outside a sphere, for the sphere is limited in size, or it can have one or more points in common with the sphere. If the distance of the line from the center is equal to a radius, it has but one point in common with the sphere (why?), and so is called *tangent to the*

sphere, the point being called the *point of contact*. If the distance of the line from the center is less than a radius, two, and but two, lines of length equal to a radius can be drawn to it from the center (determining the plane of the line and the center and using plane geometry); so the line meets the surface in two points and is called a *secant*. Show that a line tangent to a sphere is not tangent to all the circles through its point of contact.

209. A line whose distance from the center of a sphere is more than the length of a radius does not meet the surface of the sphere.

210. A line tangent to a sphere must be perpendicular to the radius drawn to its point of contact.

211. The plane perpendicular to a tangent at its point of contact passes through the center of the sphere.

197. Relative Positions of a Plane and a Sphere. A plane is *outside a sphere* if its distance from its center is more than the length of a radius. Why?

A plane is *tangent to a sphere* if it has one point in common with the sphere. The possibility of such a plane is shown in § 198. The common point is called, as in the case of the tangent line, the *point of contact*.

A plane is a *secant plane* if it cuts the sphere, the intersection with the surface necessarily (§ 101) being a closed line. See § 203.

* 198. *A plane perpendicular to a radius of a sphere at its surface extremity is tangent to the sphere.*

This proves that there can be a plane tangent to a sphere at any point on its surface. Why?

* 199. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

For the point of tangency is nearest the center. This proves that there can be but one tangent plane at a point on the surface. Why? §§ 198 and 199 show that a tangent plane is a plane whose distance from the center equals the length of the radius. Similarly a secant plane is one whose distance from the center is less than the length of the radius.

** 200. If a plane is tangent to a sphere, a perpendicular to it at the point of contact passes through the center of the sphere.*

212. All lines tangent to a sphere at a point are perpendicular to the radius drawn to that point.

213. All lines tangent to a sphere at a point are in the plane tangent to the sphere at that point.

214. Two lines tangent to a sphere at a point determine the plane tangent to the sphere at that point.

201. Circumscribed Polyhedrons. A polyhedron is said to be *circumscribed* about a sphere, and the sphere to be *inscribed* in the polyhedron, if the sphere is tangent to all the faces of the polyhedron.

** 202. One, and but one, sphere can be inscribed in any given tetrahedron.* For there is one point equidistant from its faces.

203. Theorem I. *Every plane section of a sphere is a circle whose center is the foot of the perpendicular from the center of the sphere to its plane.*

Plane sections of a sphere are called *circles of the sphere*. "Circle" is usually used for "circle of a sphere."

215. If a secant plane is gradually moved farther from the center of a sphere, describe the change in its intersection with the surface.

216. A line tangent to a sphere is tangent to every circle through the point of contact, in whose plane it lies.

204. **Axis and Poles of a Circle.** The diameter perpendicular to a circle is called its *axis*; the points where the axis cuts the surface of the sphere are called the *poles* of the circle.

205. COR. 1. *A line perpendicular to a circle at its center, or one joining the center of the circle to the center of the sphere (unless they are the same point), is the axis of the circle.*

206. COR. 2. *Sections through the center of a sphere are all equal, and are the largest circles of the sphere. Of others, two that are equidistant from the center are equal, and conversely; and, of two not equidistant from the center, the nearer is the greater, and conversely.*

207. **Great and Small Circles.** A section through the center of a sphere is called a *great circle*; any other sec-

tion is called a *small circle*. In the diagram, G is a great circle, and S is a small circle. The axis of S is PP' , its poles being P and P' . What relation to one another have r (the radius of the sphere), r' (the radius of the small circle), and d (its distance from the center)?

The circles of latitude and the meridians on the earth's surface are familiar examples of circumferences of circles of a sphere. The equator and the meridians are circumferences of great circles.

*208. *The center of a great circle is the center of the sphere.*

*209. *Any two great circles of a sphere intersect in a diameter.*

*210. *A great circle bisects the sphere and its surface.*

*211. *Two points on the surface (not extremities of a diameter) and the center of the sphere determine a great circle; any three points on the surface determine a circle.*

212. COR. 3. *Parallel circles of a sphere have the same axis and the same poles.*

The circles of latitude on the earth's surface have the axis of the earth as their axis, and its poles as their poles.

213. COR. 4. *A great circle through the poles of another circle is perpendicular to it; a great circle perpendicular to another circle contains its axis and its poles.* This is sometimes stated: *For a great circle to be perpendicular to another circle, it is necessary, and it is sufficient, that it contain its poles.*

217. A great circle that contains one of the poles of another circle must contain the other pole also.

214. COR. 5. *The locus of points in a sphere equidistant from all points on the circumference of a circle is the axis of the circle; the poles of a circle are equidistant from all points on its circumference.*

215. Polar Chords and Polar Distances. The chords from the pole of a circle to points on its circumference are called *polar chords*; the great circle arcs from its pole to points on its circumference are called *polar distances*. Unless otherwise stated the chords and distances from the nearer pole are meant.

** 216. The polar distances of the same, or of equal, circles on a sphere are equal.*

217. A Quadrant. One quarter of the circumference of a great circle is called a *quadrant* of the sphere. The degree measure of a quadrant is 90° .

** 218. The polar distance of a great circle is a quadrant, and conversely.*

Use the central angles subtended by the polar distances.

** 219. If two points on a spherical surface are a quadrant's distance apart on the great circle arc through them, each is the pole of a great circle through the other.*

Using either as a pole, draw the axis, then draw the great circle of which it is the pole.

** 220. If a point on the surface of a sphere is at a quadrant's distance from each of two other points of the surface, it is the pole of a great circle through them.*

Proceed as in § 218 or § 219. What cases are there?

221. Angles between Arcs. The *angle between two arcs* is the angle between the tangent lines at the point of intersection.

218. Two coplanar circles are perpendicular to each other when a radius of one is tangent to the other. Must a radius of each arc be tangent to the other circle?

222. Spherical Angles. The most important class of

angles between arcs is that having great circle arcs as the arms of the angle. Such an angle is called a *spherical angle*. As two great circles intersect in a diameter, the tangents are both perpendicular to this diameter, and therefore are the arms of a measuring angle of the dihedral angle between the great circles.

* 223. *The angle between two great circle arcs equals the measuring angle of the dihedral angle between their planes.*

* 224. *A great circle arc through a pole of another great circle is perpendicular to its circumference.*

225. Measurement of Spherical Angles. The most convenient way to measure a spherical angle is by the measuring angle of the corresponding dihedral angle, drawn

at the center of the sphere. This angle is evidently in the great circle perpendicular to the edge of the dihedral angle, and therefore perpendicular to the planes in which the arms of the spherical angle lie.

The spherical angle XTY equals $\angle ATB$ or $\angle XOY$, measuring angles of the dihedral angle between the planes TXT' and TYT' . But $\angle XOY$ might be measured by arc XY ; therefore it follows that

* 226. *A spherical angle is measured by the subtended great circle arc having its vertex as a pole.*

219. Explain how a figure can be drawn so that it will be bounded by three great circle arcs, each perpendicular to the others.

THE MATERIAL SPHERE

* 227. *To find the radius of a given circumference on a given material sphere.*

Select three points on the circumference, and consider a triangle, with these three points as vertices, as inscribed in the circle. Construct, in some plane, the triangle having these three sides (using an ordinary compass to carry the lengths, for it will measure a chord of a circle on a sphere just as well as a sect on a plane). Circumscribe a circle about this triangle, and it will equal the given circle. Why?

* 228. *To describe a circumference on a sphere with a given pole, and a given polar chord.*

* 229. *The polar chord of a circle of a sphere is the mean between the diameter of the sphere and its own projection upon the axis of the circle; the radius of a circle is the mean between the two sects it makes on the axis of that circle.*

Pass a great circle through the pole, and work in that plane.

230. Theorem II. *To find the diameter of a given material sphere.*

With any pole, and any polar chord that is of convenient length, describe a circumference on the sphere; find the radius of this circle, and from these two lengths, construct the right triangle considered in § 229.

220. On a sphere of known diameter, to construct a circumference such that a triangle inscribed in it shall be congruent to a given triangle.

221. If the diameter of a material sphere is known, to construct on it a circumference of given radius.

222. To construct a circumference on a given material sphere so that it will be a given distance from the center.

223. To construct parallel circumferences on a material sphere.

224. Given any circumference on a sphere, to find any number of points on the circumference of the great circle perpendicular to the given circle at any given point.

231. Relative Position of Two Spheres. Two spheres either lie entirely outside each other; lie outside each other except for one point in common; have more than one point in common, without either being entirely contained in the other; or one is entirely contained in the other. Only the second and third of these possibilities are of special interest in solid geometry.

232. Center Line. The line through the centers of two spheres is called their *line of centers*. The sect between the centers is called their *center sect*.

233. Tangent Spheres. Two spheres whose surfaces have one point in common are called *tangent spheres*, the point of tangency being called the *point of contact*. They are said to be *externally* or *internally* tangent, according as they are outside each other, or one is contained in the other. Unless otherwise stated, *tangent* will be used to mean *externally tangent*.

* **234.** *Two spheres whose surfaces meet in the center line are tangent.*

* **235.** *Two spheres tangent to the same plane at the same point are tangent.*

236. Theorem III. *If the surfaces of two spheres meet at a point not on the center line, they meet in the circumference of a circle that is perpendicular to the center line.*

If a plane is passed through the point and the two centers, the two great circles in which this plane intersects the spheres intersect in two points and have a common chord. Rotate these circles to generate the spheres, and examine the figure generated by their common points.

225. If one sphere is inside another, and their surfaces meet on the center line, the spheres are tangent.

226. If the center sect of two spheres is less than the sum, but greater than the difference of the radii, the surfaces of the spheres intersect in the circumference of a circle.

227. If the center sect of two spheres is less than the difference of the radii, how must the spheres lie?

228. If two equal spheres meet, the circle in whose circumference their surfaces intersect is equidistant from their centers.

229. Find the center of a sphere whose surface passes through the circumference of a given circle, and contains a given point not coplanar with the circle.

230. If from a point within a sphere, three chords are drawn, each perpendicular to the other two, the sum of the squares of the sects of the chords equals half the sum of the squares of the diameters of the circles determined by the chords.

SECTION II. SPHERICAL POLYGONS AND POLYHEDRAL ANGLES

237. Spherical Polygons. A portion of a spherical surface bounded by three or more arcs of great circles is called a *spherical polygon*. The bounding arcs are the *sides* of the polygon, the spherical angles between the sides are its *angles*, and the vertices of the angles are its *vertices*.

238. Convex Polygons. A spherical polygon is *convex* if no side when extended can cut the surface of the polygon. This is sometimes stated, a spherical polygon is convex if when any side is extended to form a circumference, the whole polygon lies on the surface of one of the hemispheres bounded by this circle. Unless otherwise stated, only convex polygons will be considered, and the word "convex" should be understood whenever necessary.

239. Spherical Triangles. A spherical polygon of three sides is called a *spherical triangle*. The meanings of *isosceles*, *equilateral*, *scalene*, *right-angled*, and any other such terms, are the same as when applied to plane triangles.

240. Relation between Polyhedral Angles and Spherical Polygons. (See §§ 99–102.)

Any spherical polygon is subtended by a polyhedral angle whose vertex is at the center of the sphere, for its sides are arcs of great circles whose planes intersect at the center, and so bound a polyhedral angle. Conversely,

any polyhedral angle whose vertex is at the center of a sphere subtends a spherical polygon on the surface, for its faces cut the surface in great circle arcs, which are the sides, and its edges cut the surface in points, which are the vertices, of such a polygon. $O-ABCD$, and $O-A'B'C'D'$ are polyhedral angles subtending the spherical polygons $ABCD$ and $A'B'C'D'$.

241. Central Polyhedral Angles. A polyhedral angle whose vertex is at the center of a sphere is called a *central* polyhedral angle, and the subtended spherical polygon is spoken of as its polygon.

** 242. Each face angle of a central polyhedral angle is measured by the subtended side of its spherical polygon.*

** 243. Each dihedral angle of a central polyhedral angle has the same measure as the corresponding spherical angle of its spherical polygon.*

244. Corresponding Propositions on Polyhedral Angles and Spherical Polygons. From the relations existing between the parts of a central polyhedral angle and the parts of its spherical polygon, it is evident that for every proposition concerning the face angles and the dihedral angles of a polyhedral angle, there must be a corresponding proposition concerning the sides and the angles of a spherical polygon, and conversely.

In the propositions dealing with these two kinds of figures, this relation is shown by stating the propositions in pairs, one about polyhedral angles, the other about spherical polygons. The one that is more convenient to

prove independently is stated first, the other follows from the relations shown in §§ 242 and 243.

245. Theorem IV. (a) *The sum of any two face angles of a trihedral angle is greater than the third face angle.*

(b) *The sum of any two sides of a spherical triangle is greater than the third side.*

FIRST PROOF: Let the trihedral angle be $V-ABC$. Pass a plane through CV perpendicular to plane AVB , meeting it in $C'V$. Then $C'V$ contains the projections of BV and AV upon the plane CVC' . Why?

Therefore $\angle AVC' < \angle AVC$, $\angle BVC' < \angle BVC$. Why? And $\angle AVC' + \angle BVC' < \angle AVC + \angle BVC$. But $\angle AVC' + \angle BVC' =$, or $>$, $\angle AVB$, according as VC' falls within or outside the angle AVB , so $\angle AVB < \angle AVC + \angle BVC$.

SECOND PROOF: Let the trihedral angle be $V-KLM$, and let $\angle KVL$ be its largest face angle. On this angle, cut off $\angle RVL = \angle LMV$. Cut off VX on VR , and $VC = VX$ on VM . Pass a plane through X and C cutting VK at A and VL at B . Since $\angle BVC = \angle XVB$, it is only necessary to prove that $\angle CVA > \angle AVX$ (by comparing their triangles) to obtain the required conclusion.

Explain in full why (b) follows from (a).

231. Any side of a spherical triangle is greater than the difference of the other two sides. To what proposition on polyhedral angles does this correspond?

246. COR. *If two circles on a sphere meet at a point on the great circle arc through their poles, they cannot meet again.*

For, if they did, on drawing the polar distances to that point, the sum of two sides of the triangle formed would equal the third side. Could two circles meet at a second point on the circumference of the same great circle?

247. Theorem V. *The shortest line from one point to another on the surface of a sphere is the minor arc of the great circle through them.*

Let PQ be the minor arc of a great circle through the given points P and Q ; then the shortest line from P to Q on the surface must contain every point of PQ (therefore must be PQ), for if there is even one of the points of PQ that it does not contain, it can be made still shorter.

Suppose the shortest line from P to Q does not contain T . With P and Q as poles, and PT and QT as polar distances, describe circles, which will therefore meet at T . Then the circles meet only at T , and any line from P to Q not through T must intersect the circles at different points K and L . But if circles P and Q were rotated so that K and L were at T , the line $PKLQ$ would be shortened by the length KL , its other parts remaining the same. Therefore any line not containing T is not the shortest line from P to Q .

248. Distance on a Spherical Surface. The *distance* between two points on a spherical surface means the length of the minor arc of the great circle through them, because the great circle arc is the shortest line between two points on the surface of a sphere and therefore corresponds to the straight line on a plane surface. Unless otherwise stated, only great circle arcs will be used.

249. Theorem VI. (a) *The sum of the face angles of any polyhedral angle is less than a perigon.*

(b) *The sum of the sides of any spherical polygon is less than the circumference of a great circle.*

Let $ABCDEF$ be a plane section of the polyhedral angle with vertex V . Then at A , $\angle FAV + \angle VAB > \angle FAB$ of the polygon $ABCDEF$ (why?), and similarly at each vertex of the polygon. Therefore the sum of the angles in the faces of the polyhedral angle at $A, B, C, \dots > (n - 2)$ st. angles. Why? But the sum of all the angles in the face triangles = ? Therefore the sum of the angles at V = ?

Or, join a point O within $ABCDEF$ to its vertices. The faces of the polyhedral angle contain the same number of triangles as $ABCDEF$, and so the sum of the angles is the same. But the angles in the faces at A, B, C, D, E, F are greater than the angles in the polygon. What follows?

Explain fully why (b) follows from (a). If it is preferred, (b) can be proved independently, and (a) can be deduced from it, the proof for any case following the method shown here for a quadrilateral.

To show that the sum of the sides of any quadrilateral $ABCD$ is less than a circumference, extend AB and AD to their other intersection A' . Extend DC to meet AA' at X . Now use $BC < BX + XC$; $XC + CD < XA' + A'D$, to show that the perimeter of $ABCD$ is less than $AA' + A'A$, or a circumference.

232. If, in the figure of Theorem VI, V approaches indefinitely near to the plane $ABCDEF$, what change is there in the sum of the angles at V ? what change if V moves indefinitely far away? Between what limits might it be said that the sum of the face angles of a polyhedral angle must lie?

233. Between what limits might the sum of the sides of a spherical polygon be said to lie?

234. Use the method shown in (b) of Theorem VI to prove the sum of the sides of a triangle less than a circumference; of a hexagon.

235. Prove, without using Theorem V, that the great circle arc between two points is less than any other line between those points made up of sects of great circle circumferences.

236. If from the ends of a side of a spherical triangle, great circle arcs are drawn to a point within the triangle, the sum of those arcs is less than the sum of the other two sides of the triangle.

250. Regular Polyhedrons. A polyhedron is said to be *regular* if its faces are all regular polygons of the same number of sides. Since each edge is common to two faces, this means that all the edges are equal, and that all the faces are congruent. It will be evident from the descrip-

tions of the regular polyhedrons that all the polyhedral angles of a regular polyhedron are congruent, and that all its dihedral angles are therefore equal.

251. Theorem VII. *There cannot be more than five regular polyhedrons.*

The sum of the face angles of a polyhedral angle is less than 360° . Why? Since an angle of an equilateral triangle is 60° , there can be three, four, or five such face angles. Six such angles around a point would lie in a plane, and seven or more would be impossible. The angle of a square is 90° , so there could be three such face angles. Four would lie in a plane, and five or more would be impossible. An angle of a regular pentagon is 108° , so there could be three such face angles, but no more. Three hexagons would lie in a plane, four would be impossible; and three polygons of any number of sides greater than six would be impossible. Therefore solid angles can be formed by three, four, or five equilateral triangles, three squares, or three regular pentagons, but in no other way from regular polygons.

It is possible (by following construction methods), to prove that a regular polyhedron can be constructed by using each of these possibilities. The method will be shown, and the polyhedrons will be described and named.

252. The Regular Tetrahedron. The *regular tetrahedron* is a solid bounded by four equilateral triangles. It could be constructed by erecting a perpendicular at the circumcenter of an equilateral triangle, and cutting this line by a circle around one of the vertices as a center, with the side as a radius. The point so found would be the fourth vertex of the tetrahedron, which is evidently

a triangular pyramid. It is the polyhedron having three equilateral triangles about each vertex.

As the faces of the regular tetrahedron are equilateral triangles, it is not difficult to find its area and volume in terms of the side. Given the edge E , the total area is $E^2\sqrt{3}$. To find the volume, the altitude is necessary, and it can be found by use of a right triangle; as, in the figure, $VC = E$, $OC = \frac{2}{3}$ the median or altitude, $= \frac{2}{3} \times \frac{E}{2}\sqrt{3} = \frac{E}{3}\sqrt{3}$; therefore $VO = \frac{E}{3}\sqrt{6}$, and the volume $= \frac{E^3}{12}\sqrt{6}$.

253. The Regular Hexahedron or Cube.

The polyhedron having three squares around each vertex is a *regular hexahedron*, or a *cube*. It is a regular rectangular parallelepiped.

If the edge is E , its surface is $6E^2$, and the volume is E^3 ; the diagonal of a face is $E\sqrt{2}$, and of the cube is $E\sqrt{3}$. Why?

237. If the diagonal of a cube is 8 in., find its volume.

238. What is the length of an edge of a cube, if the cube is twice as large as the cube of edge 3 in.?

239. Given two cubes, to construct a cube whose area is the sum of their areas.

NOTE. It is not possible, by using the straight edge and the compass, to add the volumes of cubes, or to multiply the volume of a cube, obtaining another cube as the result. This means that the same rule is true for the other solids, so while it is possible to construct a sphere or a regular tetrahedron such that its area shall be the sum of the areas of two figures of the same kind, it is not possible to perform the same operation on their volumes.

The problem of *duplication of the cube*, that is, of the construction (with straight edge and compasses) of a cube of volume double that

of a given cube, ranks with the squaring of the circle and the trisection of an angle as one of the geometrical impossibilities, on which mathematicians worked for centuries. Any history of mathematics will give an interesting account of these attempts.

240. To construct a cube having a given edge.

254. The Regular Octahedron. The solid having four equilateral triangles around each vertex is a *regular octahedron*. It might be described as two square pyramids whose edges are all equal, placed base to base.

If the octahedron has the edge E , its surface is $2E^2\sqrt{3}$. To

obtain the volume, drop a perpendicular from X to $ABCD$ at O . Then AXC is an isosceles triangle with sides $CX=E$, $XA=E$, $AC=E\sqrt{2}$; therefore $\angle CXA$ is a right angle (why?), and XO and AO are each $\frac{E}{2}\sqrt{2}$. The volume is $\frac{E^3}{3}\sqrt{2}$.

241. To construct the regular octahedron of edge E .

242. Find the volume of a regular octahedron of edge 5 in.

243. If the diagonal of a regular octahedron is 12 in., find its area.

244. Use the prismatoid formula to find the volume of the octahedron.

245. Show that the lines joining the midpoints of the adjoining faces of a cube are the edges of a regular octahedron, and find the ratio of its area and volume to the area and volume of the cube.

255. The Regular Dodecahedron. The solid having three regular pentagons around each vertex is a *regular dodecahedron*. It might be described as a solid having

two regular pentagons as parallel bases, the other faces being ten regular pentagons, arranged five around each of the bases.

256. The Regular Icosahedron. The solid having five equilateral triangles about each vertex is a *regular icosahedron*. It might be described as two pyramids with penta-

gons as bases, connected by ten equilateral triangles, each having for its base a side of one pentagon, and for its vertex, a vertex of the other pentagon.

Neither the dodecahedron nor the icosahedron is of much importance in elementary geometry, but they, as well as the other regular polyhedrons, are of importance in crystallography.

246. How many edges and vertices have the regular dodecahedron and icosahedron?

257. Vertical Polyhedral Angles. If the planes that bound a polyhedral angle are extended through the vertex, they bound a second polyhedral angle that is called *vertical*, or *opposite*, to the first angle. In the figure of § 240, $O-ABCD$ and $O-A'B'C'D'$ are vertical polyhedral angles.

258. Opposite Spherical Polygons. Spherical polygons subtended by vertical central polyhedral angles are called *opposite* spherical polygons. In the figure of § 240, $ABCD$ and $A'B'C'D'$ are opposite spherical polygons. If the sides of opposite spherical polygons that are subtended by vertical face angles at the center are considered as *corresponding sides*, and the spherical angles having the ends of a diameter as their vertices are considered as *corresponding angles*, it is clear that corresponding sides are arcs of the same great circle circumference, and that corresponding spherical angles are formed by the same two planes.

259. Division of the Surface by Circumferences of Great Circles. The circumferences of two great circles meet at the ends of a diameter, and divide the surface into four parts called *lunes*. (See also § 285.) A third great circumference, not through either end of the same diameter, will cut each of these circumferences twice, thus dividing each of the lunes into two triangles, and the surface of the sphere into eight triangles. These triangles are evidently four pairs of opposite triangles, since they are formed by the same great circle planes.

It follows that, if the sides of any spherical triangle are extended, they divide the surface of the sphere into eight triangles, one of which is opposite to the given triangle.

260. Symmetric Spherical Polygons and Polyhedral Angles. Two spherical polygons are called *symmetric*

when their corresponding sides are equal, and their corresponding angles are equal, *the parts being arranged in opposite order*, in the two polygons. For example, if a polygon has the sides a, b, c, d , arranged in the order indicated when read in the direction taken by the hands of a clock (*clockwise*), its symmetric polygon will have its sides in this order when read in the opposite direction, that is, *counter-clockwise*.

Similarly, polyhedral angles are *symmetric* when their corresponding parts are equal, but are arranged in opposite order.

* **261.** *Vertical polyhedral angles are symmetric.* See the figure of § 240. Show that the parts are correspondingly equal, then examine the order, viewed from the common vertex.

* **262.** *Opposite spherical polygons are symmetric.* See the figure of § 240. View the polygons from the center of the sphere; otherwise the order may become confused.

263. Coincidence of Great Circle Arcs and of Spherical Angles. The circumferences of great circles coincide if they have two points other than the ends of a diameter in common, and if they have the same center. In other words, two points on the surface of a sphere determine the circumference of a great circle. (Why?) Therefore, if one of two great circle arcs is placed along another, with one end in common, their centers being the same, it will lie along the other throughout its length. Therefore the first arc will coincide with the second if the two arcs are equal; the first will include the second if the first is the longer; and the first will be included by the second if the first is the shorter.

If one of two spherical angles is placed on another (the center of their spheres being the same), with their vertices coinciding, and one arm of the first angle along the corresponding arm of the other angle, then the other arm of the first angle must also fall along its corresponding arm if the angles are equal (why?), so making the angles coincide; it must fall outside the corresponding arm if that angle is the greater; and it must fall inside the corresponding arm if that angle is the smaller.

In placing either an arc, or a spherical angle, on another, it must be considered as being slid along the surface into the required position, for if it is turned over, that is, if the position of the center of the sphere is changed, the curvature is reversed, and the arcs are no longer on the same sphere, and cannot coincide. This is most easily shown by the arcs

of circles in a plane, the same reasoning applying for the arcs on a sphere. Let the circles O and O' be equal, and let arc AB equal arc XY . Although

they have two points in common, they do not coincide, for they are not on the circumference of the same circle. Now suppose circle O' to be superposed on circle O with the centers coinciding, and so that arc XY lies along arc AB with one end in common. Then in this position they coincide. Similarly, if arc KL equals arc AB , it coincides with it if it is supposed to slide along the circumference until L falls on A . It cannot coincide with AB if it is turned over so that its center is no longer at O .

264. Congruence and Symmetry of Spherical Polygons.

Two spherical polygons are *congruent* if they can be made to coincide. If they can coincide, their corresponding (or *homologous*) parts must be equal, and arranged in the same order. Conversely, if their corresponding parts are equal, and arranged in the same order, they can be superposed, and by examining the position of each vertex and side, they can be shown to coincide. If, however, the parts are equal, but arranged in opposite order, the figures cannot be made to coincide, that is, symmetric polygons are not congruent. (See § 272.)

If in $\triangle ABC$, $\triangle DEF$, and $\triangle GHK$, $AB = DE = GH$, $BC = EF = KG$, $CA = FD = HK$, and correspondingly for the angles, the triangles are not all congruent, for the parts of $\triangle GHK$ are arranged in opposite order to that in which the parts of $\triangle ABC$ and $\triangle DEF$ are arranged, and so $\triangle GHK$ cannot be superposed on one of them by sliding it along the surface, — as could be done with $\triangle ABC$ and $\triangle DEF$, — but must be turned over, that is, placed with the center of its sphere outside of sphere O instead of at center O , to bring its parts in the right order. But this reverses the curvature of its sides, — that is, they are no longer great circle arcs on the same sphere, — so they cannot be made to coincide, and the triangle will not coincide with either of the other triangles, and so is not congruent to them.

Note that polygons symmetric to the same polygon are congruent.

Explain why two plane triangles are congruent if their parts are equal but arranged in opposite order.

265. Congruence and Symmetry of Polyhedral Angles.

Polyhedral angles, like spherical polygons, are congruent when they can be made to coincide. They can be made to coincide when their corresponding parts are equal and arranged in the same order; but they are only symmetric when the parts are equal, but are arranged in opposite order.

Examples of symmetric bodies, that is, of bodies that are equal in all respects, but cannot coincide because their parts are arranged in opposite order, are numerous. Among the most familiar examples are one's hands, feet, ears, shoulders, a right and a left glove, a pair of shoes, the inside and the outside of a door, the two halves of a piano if cut from front to back, the two blades of a pair of scissors, etc.

266. Theorem VIII. *(a) Two triangles on the same sphere are either congruent or symmetric if they have two sides and the included angle of one respectively equal to two sides and the included angle of the other.*

(b) Two trihedral angles are either congruent or symmetric if they have two face angles and the included dihedral angle of one respectively equal to two face angles and the included dihedral angle of the other.

Superpose if the parts are arranged in like order. If not, superpose one on the triangle that is opposite to the other.

267. Theorem IX. *(a) Two triangles on the same sphere are either congruent or symmetric if they have two angles and the included side of one respectively equal to two angles and the included side of the other.*

(b) Two trihedral angles are either congruent or symmetric if they have two dihedral angles and the in-

cluded face angle of one equal to two dihedral angles and the included face angle of the other.

247. Prove Theorem IX without superposing.

268. Theorem X. *(a) In an isosceles spherical triangle, the base angles are equal.*

(b) If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

Prove as in plane geometry.

269. COR. 1. *In an isosceles spherical triangle, the bisector of the vertex angle, the median to the base, the altitude, and the perpendicular bisector of the base, are all one line.*

270. COR. 2. *Two isosceles symmetric spherical triangles are congruent.*

248. Two spherical right triangles are congruent or symmetric if they have two sides of one equal to the corresponding sides of the other.

271. Theorem XI. *(a) Two triangles on the same sphere are either congruent or symmetric if they have the three sides of one equal respectively to the three sides of the other.*

(b) Two trihedral angles are either congruent or symmetric if they have the three face angles of one respectively equal to the three face angles of the other.

272. Theorem XII. *Two symmetric spherical triangles are equivalent.*

Draw a circle through the vertices of each triangle and prove the circles equal. Draw the polar distances to the vertices of the triangles, and prove them equal. Show that

the three triangles thus formed in one of the given triangles are respectively congruent to the three triangles formed in the other triangle, — in other words, that symmetric triangles can be divided into congruent parts, and so are equivalent.

249. The median to the base of an isosceles spherical triangle divides the triangle into equivalent triangles.

250. If the opposite sides of a spherical quadrilateral are equal, either diagonal divides it into equivalent triangles; the diagonals bisect each other.

273. Polar Triangles. If for each side of a spherical triangle that one of its poles is taken that is on the same hemispherical surface as the opposite vertex (the circle of which that side is an arc being considered to form the hemispheres), and if these three points are joined by minor arcs of great circles, the resulting triangle is called the *polar triangle* of the original triangle. In the figure, if

A' , B' , C' , are the poles of BC , CA , and AB respectively, since A and A' are on the same hemispherical surface with reference to BC , B and B' are on the same hemispherical surface with reference to CA , and C and C' are on the same hemispherical surface with reference to AB , then the great circle arcs $A'B'$, $B'C'$, and $C'A'$ form the polar triangle of triangle ABC .

274. Theorem XIII. *If the first of two spherical triangles is the polar triangle of the second, the second is also the polar triangle of the first.*

How can a point be proved the pole of an arc? No construction lines need be drawn.

Two triangles, such that each is the polar of the other, are called *polar triangles*.

275. Theorem XIV. *In two polar triangles, each angle of one is measured by the supplement of the side opposite to it in the other.*

It is necessary to prove that $\angle C$ is measured by $180^\circ - A'B'$. Since C is the pole of $A'B'$, by what arc can $\angle C$ be measured? Show that this arc is $180^\circ - A'B'$ by using the fact that A' and B' are poles of the great circles through BC and CA respectively. They therefore have what polar distances to those circumferences?

251. The polar triangle of a right triangle has one side a quadrant.

252. If the angles of a triangle are 135° , 97° , and 88° , find the opposite sides of the polar triangle.

253. Show that if one side of a spherical polygon is more than 180° , the polygon is not convex.

254. If the sides of two polar triangles meet, in how many points can they intersect?

255. Two triangles, such that one side of each is a quadrant (*quadrantal triangles*), are congruent or symmetric if they have two angles of one respectively equal to two angles of the other.

276. Theorem XV. (a) *Two triangles on the same sphere are either congruent or symmetric if they have the three angles of one respectively equal to the three angles of the other.*

(b) *Two trihedral angles are either congruent or symmetric if they have the three dihedral angles of one respectively equal to the three dihedral angles of the other.*

For the sides of their polar triangles are equal (why?), so the polar triangles are congruent or symmetric. Show that the sides of the given triangles must therefore be respectively equal.

256. If two angles of a spherical triangle are equal, the opposite sides are equal. (Use the polar triangles.) State this for a trihedral angle.

277. Theorem XVI. *The sum of the angles of a spherical triangle is greater than one, and less than three, straight angles.*

Let the angles be A, B, C , the opposite sides be a, b, c , and the angles and sides of the polar triangle be A', B', C' , and a', b', c' :

then $A + B + C = 540^\circ - (a' + b' + c')$. Why?

But $a' + b' + c'$ lies between what limits?

278. Spherical Excess. § 277 proves that the sum of the angles of a spherical triangle is greater than the sum of the angles of a plane triangle. From this it follows that the sum of the angles of any spherical polygon of n sides is more than $(n-2)$ straight angles. The amount by which the sum of the angles of a spherical polygon exceeds the sum of the angles of a plane polygon of the same number of sides is called the *spherical excess* of that polygon. The spherical excess of a triangle is shown in § 277 to be between zero and two straight angles.

SECTION III. AREA ON A SPHERICAL SURFACE

279. Generation of the Surface of a Sphere. It has been shown (§ 188) that the rotation of a semicircle about its diameter generates a sphere. It follows that the surface generated by the semicircumference is a spherical surface, and from this fact the area of a spherical surface can be found.

280. Zones. That part of the surface of a sphere which is included between two parallel planes that meet the surface is called a *zone*. Either of the planes may be a tangent or a secant plane, so the zone can be bounded by two circumferences on the surface, by but one circumference when one plane is tangent, or it can include the whole surface. The perpendicular between the planes is the *altitude* of the zone.

The zones on the surface of the earth are familiar examples.

281. Generation of a Zone. The rotation of any arc of a semicircle about its diameter generates a zone, for all points of the surface so generated are equidistant from the center of the circle, and the surface lies between planes perpendicular to the diameter.

282. *If, in the generating arc of a zone, a broken line is inscribed whose vertices divide the arc into equal parts, then, as the number of these parts is increased indefinitely, the area generated by the broken line approaches the area of the zone as its limit.*

This is practically an axiom of the sphere, although it is evident from the plane geometry discussion of the regular polygon inscribed in a circle. It may be called a proposition, but in any case it is assumed here without proof.

283. Theorem XVII. *The area of a zone is equal to the product of its altitude and the circumference of a great circle. $\text{Area} = 2 \pi r h$.*

For, if a broken line is inscribed, as explained in § 282, the area generated by any one sect will be $2 \pi a h'$, where a is the apothem to the broken line, and h' is the altitude of the frustum generated by that sect. (See § 171, the perpendicular in that formula being the apothem in this figure). On adding these areas for all the sects, and representing the sum of the altitudes, or the altitude of the zone, by h , the total area generated by the broken line is $2 \pi a h$. But if the number of parts is increased indefinitely, the surface generated by the broken line approaches the zone, and the apothem approaches the radius of the sphere. Therefore the area of the zone is $2 \pi r h$.

284. COR. *The area of a spherical surface is the product of its diameter and the circumference of a great circle. $\text{Area} = 4 \pi r^2$.*

For it is a zone of what altitude?

257. Zones on the same sphere are proportional to their altitudes.

258. Find the area of a sphere of radius 2 inches. How does it compare with the area of a great circle? How does the area of any zone compare with the area of a great circle?

259. Find the area of a zone on a sphere of radius 1 ft. if the polar distance of one of its circles is 30° and of the other is 60° .

285. Lunes. That part of a spherical surface bounded by two great semicircumferences is called a *lune*. See also § 259. The lune can be of any size from zero to the entire surface of the sphere. The two semicircumferences form two spherical angles at their intersections, and these angles are equal. (Why?) In the figure, two lunes are shown, one bounded by PXP' and PYP' , the other by PYP' and PZP' ; their angles are marked A and B .

286. Addition and Subtraction of Lunes. Two lunes are *added* by placing them so that they lie on opposite sides of a common semicircumference, the entire figure so formed being their sum. They are *subtracted* by placing them on the same side of a common semicircumference, that part of the larger not contained in the smaller being their difference.

The sum or the difference of two lunes is also a lune, — since it is still bounded by semicircumferences, — as is also any multiple of a lune, for it could be formed by successive additions of equal lunes.

287. Equality of Lunes. If, on the same spherical surface, one lune is placed on another with one bounding semicircumference in common, the other circumferences either coincide, or one of them lies entirely within the lune bounded by the other. There is, therefore, no difference between equivalence and congruence of lunes, and the word *equal* will be used for them.

288. Notation for Lunes. In dealing with lunes, the lune is designated by the letter L with a subscript denoting the spherical angle at the point of intersection of the

semicircumferences; for example, L_A denotes the lune whose spherical angle is A ; L_{180° denotes the lune whose spherical angle is 180° ,—in other words, a hemispherical surface. That this is a satisfactory way in which to denote a lune will be evident from the following paragraphs.

289. Relations between Lunes and their Spherical Angles.

From the foregoing discussion, and the fact that a spherical angle has the same measure as the dihedral angle between the circles whose arcs are its arms, the following propositions about lunes follow.

* 290. *Lunes are equal if their spherical angles are equal.*

$$L_A = L_B \text{ if } A = B. \quad \text{Superpose.}$$

* 291. *If two lunes are equal, their spherical angles are equal.*

$$\text{If } L_A = L_B, \text{ then } A = B.$$

* 292. *The spherical angle of the sum, or of the difference, of two lunes is the sum, or the difference, respectively, of their spherical angles.*

$$L_A \pm L_B = L_{(A \pm B)}.$$

* 293. *The spherical angle of any multiple of a lune is the same multiple of its spherical angle.*

$$nL_A = L_{nA}.$$

* 294. *Lunes are proportional to their spherical angles.*

$$\frac{L_A}{L_B} = \frac{A}{B}.$$

Use the ordinary method of proof for the commensurable case,

295. Theorem XVIII. *The area of a lune is the same part of the area of the sphere that its spherical angle is of 360° .*

$$L_A = \frac{A}{360^\circ} (4\pi r^2).$$

260. Find the area of a lune of angle 20° on a sphere of radius 18 inches.

261. Find the angle of a lune that is one fifth of the spherical surface.

262. Find the angle of a lune that is equivalent to a trirectangular triangle (one whose angles are all right angles).

263. A lune on a sphere of radius 10 in. is equivalent to the whole surface of a sphere of radius 4 in. What is its angle?

296. Theorem XIX. *A spherical triangle is equivalent to a lune whose angle is one half the spherical excess of the triangle.*

Call the triangle ABC , T . Let its spherical excess be E , and letter the intersections of the great circumferences that form its sides as in the figure. Then the semisurface $AC'A'C$ (or L_{180°) is composed of four triangles, whose areas are as follows:

$$\triangle ABC = T; \triangle AC'B = L_C - T; \triangle A'CB = L_A - T;$$

$$\triangle A'BC' = \triangle AB'C \text{ (opposite triangles), } \therefore = L_B - T.$$

Therefore on adding these values of the four triangles,

$$L_{180^\circ} = L_A + L_B + L_C - 2T, \text{ and on solving for } T,$$

$$T = L_{\frac{180^\circ - A - B - C}{2}} = \frac{L_E}{2}.$$

297. COR. 1. *The area of a spherical triangle is the same part of the area of the sphere that its spherical excess is of 720° .*

$$T = \frac{E}{720^\circ} (4\pi r^2).$$

298. COR. 2. *The area of a spherical polygon is the same part of the area of the sphere that its spherical excess is of 720° .*

$$P = \frac{E}{720^\circ} (4\pi r^2).$$

299. Notation for Spherical Triangles and Spherical Polygons. In finding the areas of spherical polygons, since only the angles are used, a convenient notation is to denote a triangle by T with a subscript to show its angles. Thus, $T_{(A, B, C)}$ would represent the triangle of angles A , B , and C , and $T_{(97^\circ, 85^\circ, 103^\circ)}$ would represent the triangle having angles of 97° , 85° , and 103° . It is convenient also to denote a polygon of more than three sides by P with a subscript to show its angles.

264. On a sphere of radius 10 ft., find the area of (a) a triangle of angles $84^\circ 30'$, $111^\circ 28'$, $35^\circ 17'$; (b) a pentagon each angle of which is 120° .

265. If an equiangular triangle is one tenth of the spherical surface, find its angles.

266. A spherical triangle whose angles are in the ratios 1:2:3 is equivalent to a zone of altitude $\frac{r}{3}$. Find its angles.

267. A spherical polygon on a sphere of radius 6 in. has each angle 160° , and its area is 44 sq. in. How many sides has it?

268. Find the area of $T_{(39^\circ, 38^\circ, 179^\circ)}$ in terms of r .

269. Find the area of $P_{(168^\circ, 178^\circ, 107^\circ, 123^\circ 30', 177^\circ 14', 154^\circ 16')}$ on a sphere of radius 1 ft.

SECTION IV. VOLUME OF A SPHERE AND ITS PARTS

300. One method of finding the volume of a sphere is to consider it the limit approached by a circumscribed polyhedron as the number of its faces is increased indefinitely. This corresponds to the method used to find the area of a circle. As in dealing with the circle, there must be assumed a statement that the figure approaches a limit as the number of its faces is increased.

If each vertex of a circumscribed polyhedron is joined to the center of the sphere, and planes are passed tangent to the sphere at the points where these lines meet the surface (thus forming, with the intercepted parts of the faces of the original polyhedron, a new polyhedron of a larger number of faces), the area and the volume of the polyhedron approach the area and the volume of the sphere as limits, as the number of faces is increased indefinitely.

301. Theorem XX. *The volume of a sphere is one third the product of its radius and its area.*

$$V = \frac{4}{3} \pi r^3.$$

FIRST METHOD. The volume of a circumscribed polyhedron is one third its area times the radius (why?). Increase the number of faces indefinitely.

SECOND METHOD. Show that a sphere of radius r , and a cylinder of radius r and altitude $2r$ that has been hollowed out from each base to the center in the form of right circular cones of radius r and altitude r , are Cavalieri

bodies, since the section of each at a distance d from the center is $\pi r^2 - \pi d^2$. Then find the volume of the hollowed cylinder.

302. Solids Extending from the Center of the Sphere to its Surface. The first method of proof is applicable to all solids that extend from the center of the sphere to some part of its surface, if they are bounded laterally by planes or conical surfaces, for all such solids can be considered as limits of the sum of an indefinitely large number of pyramids with vertices at the center of the sphere, and bases tangent to the sphere.

303. Spherical Cones and Spherical Sectors. That part of a sphere included in a conical space whose vertex is at the center of the sphere is called a *spherical cone* (as $O - B_1$).

It could be generated by revolving a plane sector about one of its bounding radii as an axis, so it is considered as one kind of *spherical sector*. That part of a sphere generated by revolving a plane sector about a diameter is called a *spherical sector* (as $O - B_2$); if it revolves about a diameter other than one of its own boundaries, it is evidently the

difference of spherical cones. The base of any kind of spherical sector is evidently a zone. The spherical cone $O-B_1$ is cut out of sphere O by the conical space $O-CD$.

* 304. *The volume of a spherical cone or a spherical sector is one third its base times the radius of the sphere.*

$$V = \frac{2}{3} \pi r^2 h$$

(where h is the altitude of the zone).

270. Express the formula for the volume of a spherical cone in terms of the distance of the plane cutting off the zone from the center of the sphere, instead of in terms of h .

271. Express the formula for the volume of a spherical cone in terms of the radius of the circle instead of in terms of h .

272. Find the volume of a spherical sector in a sphere of radius 12 in. if the radii of the two circles of its base are 6 in. and 8 in.

305. Spherical Pyramids. That part of a sphere included in a pyramidal space whose vertex is at the center of the sphere is called a *spherical pyramid*. The base of a spherical pyramid is evidently a spherical polygon. The spherical pyramid, like any other pyramid, can be triangular, quadrangular, etc. $O-ABC$ is a triangular spherical pyramid.

* 306. *The volume of a spherical pyramid is one third its base times the radius of the sphere.*

$$V = \frac{r}{3} \cdot \frac{E}{720^\circ} (4 \pi r^2) = \frac{E}{720^\circ} \left(\frac{4}{3} \pi r^3 \right),$$

where E is the spherical excess of the base.

273. State the volume of a spherical pyramid in form corresponding to that used for the area of a spherical triangle in § 297.

307. The Spherical Wedge. That part of a sphere between two great semicircles is called a *spherical wedge* or an *ungula*. Its curved surface is evidently a lune.

* **308.** *The volume of a spherical wedge is one third its base times the radius of the sphere.*

$$V = \frac{r}{3} \cdot \frac{A}{360^\circ} (4\pi r^2) = \frac{A}{360^\circ} \left(\frac{4}{3} \pi r^3 \right),$$

where A is the angle between the circles.

274. State the volume of a spherical wedge in form corresponding to that used for a lune in § 295.

309. Spherical Segments. The part of a sphere that lies between parallel planes is called a *spherical segment*. The planes can both be secant planes, one can be a tangent plane, or, in the limiting case of the whole sphere, both can be tangent planes. A spherical segment will have one base or two bases according as it is included between a secant plane and a tangent plane, or between two secant planes. The spherical surface of a spherical segment is a zone.

310. Volume of a Spherical Segment. The second method of proof used for finding the volume of a sphere is applicable also to spherical segments, for the spherical segment and the corresponding part of the hollowed out cylinder are Cavalieri bodies, and the volume of the hollowed cylinder can be found by the formulas for the cylinder and the cone. The volume of a spherical segment of one base is expressed by the formula $V = \frac{\pi h^2}{3} (3r - h)$, and that of two bases is expressed by the formula

$$V = \frac{\pi h}{6} (3r_1^2 + 3r_2^2 + h^2),$$

where h is the altitude, and r_1 and r_2 are the radii of the bases. These formulas, especially the second, are not easy to deduce, and are not necessary for calculation, as the Prismatoid Formula applies to spherical segments, and is more convenient since it can be used in both of these cases as well as for the other solids already noted.

275. A sphere is inscribed in a cylinder. Find the ratios of their areas and of their volumes.

276. A cylindrical pail of radius 10 in. is full of water. If a sphere of radius 12 in. is set in the top of the pail, how much water is spilled, counting 231 cu. in. to the gallon?

277. A sphere of radius 10 in. is hollowed out so as to leave a hollow bounded by the lateral surface of the frustum of a right circular cone of radii 6 in. and 8 in. with a diameter as axis. Find the volume of the remaining solid.

278. A sphere of 10 in. radius is hollowed out so as to leave a hollow bounded by a conical surface whose vertex is at one end of the diameter that is the axis of the cone. If the radius of the base of the cone is 6 in., find the volume of the remaining figure.

279. In building some decorations, certain trirectangular corners were filled in with blocks terminated by spherical surfaces of radius 3 in. and having the corners as centers. What was the spherical area and the volume of each block?

280. If the number of square units in the area of a sphere is equal to three times the number of cubic units in its volume, find the radius.

281. A plane passes through a sphere so that it cuts off one third the surface. What part of the volume does it cut off?

282. The rays of light from a point illumine one fourth of the surface of a sphere of radius 4 ft. How far is the point from the surface?

283. The volume of the moon is approximately $\frac{1}{49}$ that of the earth. What is the diameter of the moon if that of the earth is 7916 miles?

284. What is the radius of a sphere such that the number of cubic units in its volume equals the number of square units in the area of a great circle? such that it equals the number of linear units in the circumference of a great circle?

285. What is the area of a sphere whose area contains the same number of square units that there are linear units in the circumference of a great circle?

286. How many marbles one half inch in diameter can be made by melting a sheet of glass 4 ft. by 3 ft. and $\frac{3}{4}$ in. thick?

287. Show that the volume of a sphere inscribed in a cube of edge E is approximately .7854 E^3 .

288. Show that the volume of a cube inscribed in a sphere of radius r is approximately 1.6396 r^3 .

289. Which is the better bargain, apples at 20¢ a dozen, or apples one and a half times as great in diameter at 60¢ a dozen?

290. The diameter of the sun is $111\frac{1}{2}$ times that of the earth. How does its volume compare with that of the earth? its area?

291. If the diameter of the moon is 2000 miles, and that of the earth 8000 miles, how far from the surface of the earth would the moon have to be to shine on one fourth of the earth's surface at a time?

292. Using the same diameters as in the last exercise, find on what part of the earth's surface the moon shines if the distance between the centers of the earth and the moon is 238,000 miles.

293. What is the volume of a sphere if a circle of radius 4 in. has a polar distance of 45° ? of 30° ? of 60° ? of 90° ?

294. If a spherical triangle of angles 90° , 85° , and 125° has an area of $\frac{25}{8}\pi$ sq. in., what is the volume of the spherical pyramid on that base?

295. If the volume of a spherical wedge of 72° is $\frac{36}{5}\pi$, find the area of a lune of 72° on the same sphere.

311. The Degree of Area and Volume Formulas. In plane geometry, it has been found that the formulas for the areas of all plane figures are of second degree in terms of lengths of sects, and that the areas of similar figures are proportional to the squares of corresponding sects.

Similarly, the formulas for the areas of all solids studied are of second degree, either involving the second degree of one length,—as, the area of a sphere equals $4\pi r^2$, or the product of two lengths,—as, the lateral area of a cylinder equals $2\pi rh$. (Note that π is a number, and so does not affect the degree.)

The formulas for the volumes of all the solids studied are of third degree, either involving the cube of one length,—as, the volume of a sphere equals $\frac{4}{3}\pi r^3$; the square of one length times another length,—as, the volume of a right circular cone equals $\frac{1}{3}\pi r^2 h$; or the product of three lengths,—as, the volume of a rectangular parallelepiped equals the product of its three dimensions.

312. Similar Solids. Polyhedrons are said to be *similar* when they are bounded by the same number of correspondingly similar faces, and their corresponding polyhedral angles are equal. It is evident that any one pair of corresponding sects would be proportional to any other pair of corresponding sects. Right circular cones or cylinders are called *similar* if their axes are proportional to the radii of their bases. Any two spheres are similar, in that any pair of corresponding sects are proportional to any other pair of corresponding sects. The proportionality of corresponding sects shown in all of these definitions is the idea that is of the most importance in elementary geometry. The definitions are sometimes summed up by saying that any two solids are similar if they can be placed so that any two corresponding points will cut a sheaf of lines in the same ratio as any other pair of corresponding points. Show that solids fulfilling the conditions of this definition also satisfy the conditions of the other definitions.

* **313.** *The areas of similar solids are proportional to the squares of any pair of corresponding sects.*

* **314.** *The volumes of similar solids are proportional to the cubes of any pair of corresponding sects.*

296. If two similar solids have the area ratio 4, what is their volume ratio?

297. What relation have the volume ratio, the area ratio, and the line ratio, of two similar solids?

298. Name six kinds of solids such that any two of the same kind are similar.

299. If the volumes of two similar solids are 3141.6 cu. ft. and 25,132.8 cu. ft., and the area of the first is 1256.64 sq. ft., what is the area of the second?

300. Given a material sphere, construct the radius of a sphere having twice as great an area; any given number of times as great an area.

301. On different spheres, zones of the same altitude are proportional to the radii of the spheres. In order that zones on different spheres may fulfill the conditions of the general definition of similarity, what would have to be true of their altitudes?

302. Given a regular tetrahedron, construct another having half as great an area. What is true of their volumes?

315. SUMMARY OF PROPOSITIONS

I. DETERMINATION:

- of a sphere, by four points (194);
- of a great circle, by the center and two points on the surface (211);
- of a small circle, by three points on the surface (211).

II. RELATIVE POSITIONS; TANGENTS AND SECANTS:

(1) A point and a sphere:

The point is within, on the surface, or outside of a sphere, according as its distance from the center is less than, equal to, or greater than a radius (193).

(2) *A line and a sphere:*

The line is a secant, a tangent, or does not meet the sphere, according as its distance from the center is less than, equal to, or greater than a radius (196).

(3) *A plane and a sphere:*

The plane cuts the sphere in a circle (203), is tangent to it (198), or does not meet it (197), according as its distance from the center is less than, equal to, or greater than, a radius, and conversely.

The perpendicular to a tangent plane at its point of contact passes through the center of the sphere (200).

(4) *Two spheres*

are tangent if the center sect equals the sum of their radii (or if they meet on the line of centers) (234);

are tangent if they are tangent to the same plane at the same point (235);

intersect in a circle if they meet at a point not on the line of centers (236).

III. CIRCLES OF A SPHERE:

(1) *The great circle:*

Its center is the center of the sphere (208).

It is the largest circle of a sphere (206).

Two great circles on the same sphere intersect in a diameter (209).

The great circle bisects the sphere and its surface (210).

The great circle is determined by the center and two points on the surface (211).

The great circle is perpendicular to a circle if it contains one of its poles, and conversely (213).

Its arc is perpendicular to a circumference of a great circle if it contains one of its poles (224).

Its polar distance is a quadrant, and conversely (218).

The shortest line on the surface between two points is the minor great circle arc (247).

(2) *Axes and poles:*

The line perpendicular to a circle at its center, or joining its center to the center of the sphere, is its axis (205).

Parallel circles have the same axis and poles (212).

A pole is equidistant from all points on the circumference (214).

Polar distances of the same or of equal circles are equal (216).

Either of two points a quadrant's distance apart is the pole of a great circle through the other (219).

A point at a quadrant's distance from each of two other points is the pole of a great circle through them (220).

A polar chord is the mean between the diameter and its own projection upon the axis (229).

The radius of a circle is the mean between the sects of its axis (229).

(3) *Miscellaneous:*

Circles equidistant from the center of a sphere are equal, and conversely (206).

Circles not equidistant from the center are unequal, the one farther from the center being smaller, and conversely (206).

If two circumferences meet at a point on the great circle circumference through their poles, they cannot meet again (246).

IV. PROPERTIES OF THE SPHERE:

Radii or diameters of the same or of equal spheres are equal (190, 191).

Spheres of equal radii are congruent (191).

One, and but one, sphere can be circumscribed about a tetrahedron (195).

One, and but one, sphere can be inscribed in a tetrahedron (202).

Its surface is the locus of points at a fixed distance from a given point (its center) (192).

V. MEASUREMENT OF A SPHERICAL ANGLE:

- by the measuring angle of the dihedral angle between the planes in which its arms lie (223);
- by the subtended part of the great circle circumference of which it is a pole (226);
- when it is an angle of a triangle, by the supplement of the opposite side of the polar triangle (275).

VI. POLYHEDRAL ANGLES AND SPHERICAL POLYGONS:

(1) *Relations between polyhedral angles and spherical polygons:*

- The face angles of a central polyhedral angle are measured by the subtended sides of its spherical polygon (242).
- The dihedral angles of a central polyhedral angle are measured by the same measuring angle as the corresponding angles of its spherical polygon (243).

(2) *Properties of a spherical polygon (polyhedral angle).*

- The sum of the sides (face angles) of a spherical polygon (polyhedral angle) is less than a circumference (perigon) (249).
- In a spherical triangle (trihedral angle) the sum of the spherical angles (dihedral angles) is more than one, and less than three, straight angles (277).
- In a spherical triangle the sum of two sides (face angles) is greater than the third (245).
- In an isosceles spherical triangle (trihedral angle) the base angles (opposite dihedral angles) are equal (268).
- In an isosceles spherical triangle the bisector of the vertex angle, the median to the base, the altitude to the base, and the perpendicular bisector of the base, are all the same line (269).

(3) *Congruence and symmetry:*

- Vertical spherical polygons (polyhedral angles) are symmetric (261, 262).
- Spherical triangles (trihedral angles) are congruent or symmetric if the following parts of one are equal to the corresponding parts of the other:

- (a) two sides and the included angle (two face angles and the included dihedral angle) (266);
 - (b) two angles and the included side (two dihedral angles, and the included face angle) (267);
 - (c) three sides (three face angles) (271);
 - (d) three angles (three dihedral angles) (276);
- Isosceles symmetric spherical triangles (trihedral angles) are congruent (270).
- Symmetric spherical triangles are equivalent (272).

(4) *Polar triangles* :

If one triangle is the polar of a second, the second is also the polar of the first (274).

An angle of one of the two polar triangles and the opposite side of the other have supplemental measures (275).

VII. LOCUS OF POINTS :

The locus of points at a fixed distance from a given point is the spherical surface with the point as center and the distance as radius (192).

The locus of points in a sphere equidistant from all points on the circumference of a circle is its axis (214).

VIII. THE MATERIAL SPHERE :

To describe a circumference with a given pole and a given polar chord (228).

To find the radius of a given circumference (227).

To find the diameter of a given sphere (230).

IX. LIMITING CASES :

The zone is the limit approached by the area generated by a broken line of equal sects inscribed in the generating arc of the zone (282).

The area and the volume of a sphere are the limits approached by the area and the volume of a circumscribed polyhedron (300).

X. REGULAR POLYHEDRONS AND SIMILAR SOLIDS:

There can be but five regular polyhedrons (251).

The areas of similar solids are proportional to the squares of corresponding sects (313).

The volumes of similar solids are proportional to the cubes of corresponding sects (314).

XI. LUNES:

Lunes are equal if their angles are equal, and conversely (290, 291).

Adding or subtracting lunes adds or subtracts their angles (292).

Multiplying a lune multiplies its angle by the same factor (293).

Lunes are proportional to their angles (294).

A lune is the same part of a spherical surface that its angle is of a perigon, or 360° (295).

XII. AREA ON A SPHERE:

(1) of the sphere $= 4\pi r^2$ (284);

(2) of a zone $= 2\pi rh$ (283);

(3) of a lune $= \frac{A}{360^\circ} (4\pi r^2)$, where A is its angle in degrees (295);

(4) of a spherical triangle $=$ that of a lune whose angle is one half its spherical excess, or $\frac{E}{720^\circ} (4\pi r^2)$ (296, 297);

(5) of a spherical polygon $= \frac{E}{720^\circ} (4\pi r^2)$ (298).

XIII. VOLUME OF A SPHERE AND ITS PARTS:

(1) of the sphere $= \frac{1}{3}$ area $\times$ radius $= \frac{4}{3}\pi r^3$ (301);

(2) of the spherical cone (304), spherical sector (304), spherical pyramid (306), spherical wedge (308), $= \frac{1}{3}$ radius $\times$ area of its spherical base $= \frac{1}{3}br$;

(3) of a spherical segment. Use the Prismatoid Formula (310).

316.

ORAL AND REVIEW QUESTIONS

State the theorems in this book about spherical triangles that correspond to theorems about plane triangles. What important difference is there between the angles of a plane triangle and those

of a spherical triangle? Why is this so? How many points determine a sphere? Why? How many points on the surface determine a great circle? Why? a small circle? Why are circles through the center called great circles? Why are circles equidistant from the center equal? Define axis, pole, polar distance. What is the polar distance of a great circle? In what form is the converse of this fact used? How is the diameter of a material sphere found? In a sphere of radius 10 in., how far from the center is a circle of 6 in. radius? What is its polar chord? Which methods of proving triangles congruent or symmetric use superposition? What are the other methods? Can one triangle be polar to a second, and the second be polar to a third? Explain fully which poles are taken in forming a polar triangle. What numerical relation holds between parts of two polar triangles? If the three sides of one of two polar triangles are 97° , 83° , 69° , find as many parts of the other triangle as possible.

How is the formula for the area of a sphere obtained? What part of the surface has its formula derived in the same way? How is the area of a lune found? of a spherical triangle? of a spherical polygon? What group of solids has its volumes expressed by the formula $\frac{br}{3}$, and for what does b stand in each case? What other parts of a sphere are there, and how are their volumes found? What is the intersection of a sphere and a plane? Why? What is the intersection of two spheres? Why? When is a plane tangent to a sphere? When are two spheres tangent to each other?

Of what is a spherical surface the locus? What is the locus of points at a fixed distance from a point, and at the same time equidistant from that point and another point? at a fixed distance from each of two given points? What is the only distinction between congruence and symmetry of spherical triangles? If a great circle is perpendicular to a small circle, what must it contain? Is this sufficient to make it a perpendicular? What is known about the sum of the face angles of a polyhedral angle? What spherical theorem follows from this? What is known about the face angles of a trihedral angle? What spherical theorem follows? What are the limits of the sum of the sides of a spherical polygon? of the sum of the angles of a spherical triangle? How many equilateral triangles can be used to form a solid angle? Why? What are the regular polyhedrons and why are they the only ones? State the area and volume formulas for the sphere and all its parts. How many of the volume formulas can

be derived from the Prismatoid Formula? Which could be proved by the use of Cavalieri's Theorem, and how? Upon what power of the lengths of sects do area formulas depend? volume formulas? If the solids are similar, what rule holds for the area and the volume? If the line ratio is 3, what are the area and volume ratios? If the volume ratio is 3, what are the line and area ratios? If the area ratio is 3, what are the line and volume ratios? State all the ways in which spherical angles can be measured. What relation is there between polyhedral angles and spherical polygons? Explain why. How is this relation used? What is meant by "distance" on a spherical surface, and why? Explain how a spherical triangle can have three right angles, and tell what its polar triangle is. How are symmetric triangles proved equivalent?

GENERAL EXERCISES

303. If the strength of material is in proportion to its cross section, what is the effect of doubling the diameter of a wire? If a wire of $\frac{1}{8}$ in. radius will support an iron ball of radius 2 in., how large a wire will support a ball of radius 4 in.? What effect has doubling the diameter of a tree on the amount of wood in it?

304. How many tangents can be drawn from an external point to a sphere? How could one be constructed? How could a plane be drawn from an external point tangent to a sphere?

305. Find the locus of the centers of all spheres that are tangent to a given plane at a given point.

306. Find the locus of the centers of all spheres of given radius that are tangent to a given plane.

307. Find the locus of the centers of all spheres that are tangent to the faces of a trihedral angle.

308. A cylindrical hole of radius 6 in. was bored through a sphere of radius 10 in., the axis of the cylinder being a diameter of the sphere. What was the weight of the remaining part of the sphere, if its entire weight was 35 lb.? What ratio had the total area of the remaining figure to the area of the sphere?

309. To inscribe a cube in a given sphere.

310. To inscribe a regular tetrahedron in a given sphere.

311. To inscribe a regular octahedron in a given sphere.

312. How large an iron shell 1 in. thick would a sphere of radius 3 in. make if melted and recast?

313. A cup is in the form of a segment of a sphere of internal radius 3 in., with a flat base of radius 2 in., and a cylindrical top of radius $2\frac{1}{2}$ in., and height 2 in. How much water will it hold? (231 cu. in. = 1 gal.)

314. Why is a plumb line, in general, perpendicular to the surface of the earth?

315. In mixing paint, it is important to know of what sizes to grind the spherical particles of coloring matter in order to cover the surface evenly. If spheres of radius r are tangent to each other and to a plane, find what size other spheres must be so that they can lie between each three of those spheres and be tangent to them and to the plane.

316. If spheres of radius r are piled up in pyramidal form so that each one rests on three others that are tangent to one another, show that the centers of the outside spheres lie in the faces of a regular tetrahedron. Find the height from the highest point to the ground if cannon balls of radius 6 in. are piled three deep.

317. If the edge of a cube is doubled, what is the effect on its area? on its volume? If the area is doubled, what is the effect on its edge? on its volume? If its volume is doubled, what is the effect on its edge? on its area?

318. Prove that the shortest sect from a given point to the surface of a given sphere is along a line to the center. (Two cases.)

319. Circles are inscribed in, and circumscribed about, an equilateral triangle. Find the ratios of the areas and the volumes of the solids generated by revolving the triangle and the circles about an altitude of the triangle as an axis.

320. If a right circular cylinder of altitude equal to its diameter and a right circular cone whose slant height equals its diameter are circumscribed about a sphere, show that the total area of the cylinder is the mean proportional between the areas of the cone and the sphere, and that the volume of the cylinder is the mean proportional between the volumes of the cone and the sphere.

321. How many square inches of gold leaf are required to gild a sphere 6 ft. in diameter, no allowance being made for waste?

322. A manufacturer of marbles uses a sheet of white glass 3 ft. square and 1 in. thick, a sheet of green glass in the form of a cylindrical surface of element 2 ft., thickness $\frac{3}{4}$ in., and length along a line perpendicular to the elements 28 in., and a piece of yellow glass which when placed in a tank 3 ft. deep that is a circular frustum of radii 2 ft. and 3 ft., raises the water from 1 ft. deep to 2 ft. 3 in. deep. How many marbles $\frac{3}{4}$ in. in diameter can be made by mixing these glasses, and what proportion of each kind of glass will be in each?

323. If all possible lines are drawn from a point to a sphere, the product of two sects from the vertex to the surface on any one secant equals the product of the sects from the vertex to the surface on any other secant, and any tangent is the mean proportional between the sects of any secant.

324. To construct a sphere of given radius tangent to a given plane at a given point.

325. Find the locus of a point such that the ratio of its distances from two given points is constant.

326. In filling a measure with fruit or vegetables of approximately spherical shape, would a larger quantity be obtained with those of lesser radius, or with those of greater radius? Why?

327. If cannon balls of 1 ft. diameter are piled four deep so that each one rests on four others that are tangent to each other, find the height of the pile.

328. A cylindrical fire extinguisher is 8 in. in diameter and 2 ft. long with a spherical segment 3 in. high on its end as a great circle. How much liquid is required to charge it? How long will it take to empty it by a hose $\frac{1}{4}$ in. in diameter if the liquid flows through the hose at the rate of 25 ft. per second?

GENERAL SUMMARY

OF THE

FORMULAS OF SOLID GEOMETRY

317. MEANING OF THE LETTERS USED IN THE FORMULAS

h = altitude, the perpendicular from the vertex to the base, or between the two bases.

s = the slant height in a regular pyramid or its frustum, the element in a cylinder, and in a right circular cone or its frustum, the edge in a prism.

b, p, r : when a figure has but one base, b is its area, p is its perimeter, and, if it is circular, r is its radius.

b_1 and b_2, p_1 and p_2, r_1 and r_2, m : when a figure has parallel bases b_1 and b_2 are their areas, p_1 and p_2 are their perimeters, and, if they are circular, r_1 and r_2 are their radii; m is the area of the midsection.

p_{rt} and r_{rt} are the perimeter and radius of a right section.

I. AREA.

(1) *Lateral Area.*

(a) Frustum of a pyramid = $\frac{s(p_1 + p_2)}{2}$ (derived from trapezoid).

Applies to the frustum of a regular pyramid, the entire pyramid ($p_2 = 0$), the frustum of a right circular cone, the entire cone ($p_2 = 0$), the right prism and the right circular cylinder ($p_1 = p_2$)

For figures with circular bases, it becomes $\pi s(r_1 + r_2)$

(b) Pyramid = $\frac{s \cdot p}{2}$ (derived from the triangle).

Applies to the regular pyramid and the right circular cone.

For the cone it becomes πrs .

(c) Prism = sp_{rt} (derived from the parallelogram).

Applies to the prism (including parallelepiped), and cylinder.

For the cylinder it becomes $2\pi r_{rt}s$.

(2) *Area on a Sphere.*(a) Sphere = $4 \pi r^2$ (rotating a semicircle about its diameter).(b) Zone = $2 \pi r h$ (rotating an arc about its diameter).(c) Lune = $\frac{A}{360^\circ} (4 \pi r^2)$ where A is the angle of the lune
(derived from its ratio to the surface of the sphere).(d) Spherical polygon = $\frac{E}{720^\circ} (4 \pi r^2)$ where E is the spherical excess of the polygon (derived from the fact that a spherical triangle is equivalent to half the lune of its spherical excess).

II. VOLUME.

(1) $Prismatoid = \frac{h}{6} (b_1 + b_2 + 4 m)$

Applies to any solid whose bases are in parallel planes, if it is bounded laterally by planes, or by curved surfaces generated by revolving a straight line sect or an arc of a circle about an axis.

Applies to the prismatoid, prism, cylinder, pyramid ($b_2 = 0$), cone ($b_2 = 0$), frustum of a pyramid, frustum of a cone, sphere ($b_1 = b_2 = 0$), spherical segment, spherical wedge ($b_1 = b_2 = 0$).

Does not apply to spherical cone, spherical sector, spherical pyramid.

(2) $Frustum\ of\ a\ Pyramid\ or\ a\ Cone = \frac{h}{3} (b_1 + b_2 + \sqrt{b_1 b_2})$

(derived from the difference of two pyramids).

Applies to the frustum of a pyramid, the entire pyramid ($b_2 = 0$), the frustum of a circular cone, the entire cone ($b_2 = 0$), a prism and a circular cylinder ($b_1 = b_2$).For circular figures it becomes $\frac{\pi h}{3} (r_1^2 + r_2^2 + r_1 r_2)$.

(3) $Pyramid\ or\ Cone = \frac{bh}{3}$

(a triangular prism is composed of three equivalent triangular pyramids).

For the cone it becomes $\frac{\pi r^2 h}{3}$.

(4) $Prism\ or\ Cylinder = bh$

(derived from the rectangular parallelepiped).

For the cylinder it becomes $\pi r^2 h$.

(5) *Sphere and its Parts.*

All parts from the center to the surface, bounded laterally by planes or conical surfaces, $= \frac{br}{3}$ where b is the spherical surface and r is the radius of the sphere

(derived by taking the limit approached by pyramids from the center to the faces of a circumscribed polyhedron).

Applies to the sphere, spherical cone, spherical sector, spherical pyramid, spherical wedge.

(a) sphere $= \frac{4}{3} \pi r^3$;

(b) spherical cone or sector $= \frac{2}{3} \pi r^2 h$;

(c) spherical pyramid $= \frac{E}{720^\circ} \left(\frac{4}{3} \pi r^3 \right)$, where E is the spherical excess of the base;

(d) spherical wedge $= \frac{A}{360^\circ} \left(\frac{4}{3} \pi r^3 \right)$, where A is the angle between the bounding semicircles.

Does not apply to spherical segments.

(6) *Spherical Segment.* Use the Prismatoid Formula; or

(a) Of one base $= \frac{\pi h^2}{3} (3r - h)$.

(b) Of two bases $= \frac{\pi h}{6} (3r_1^2 + 3r_2^2 + h^2)$.

COLLEGE EXAMINATION QUESTIONS

(Chosen from recent examinations of the College Entrance Examination Board, the New York State Regents, and various colleges.)

329. A cross section of a tunnel is a rectangle 8 ft. by 12 ft., surmounted by a semicircle whose diameter is the smaller dimension of the rectangle. If the tunnel is three fourths of a mile long, how many cubic yards of earth were removed? Take π to two decimal places.

330. How many cubic feet of masonry will it take to build a dome in the form of a hemisphere, whose outer diameter is 21 ft., and whose thickness is 15 in.?

331. How should you find the center of a sphere that is to be inscribed in the regular tetrahedron $ABCD$?

332. What is the weight of a brick cupola in the form of a hemispherical shell, 6 in. thick and 15 ft. in outer diameter, if the weight per cubic ft. is 50 lb.?

333. A lateral edge of a regular pyramid measures 12 in., and each edge of its square base measures 4 in. Find its total surface.

334. A straight line perpendicular to one of two parallel planes is perpendicular to the other also.

335. A right circular cylinder 6 ft. in diameter is equivalent to a right circular cone whose base is 7 ft. in diameter. If the height of the cone is 8 ft., what is the height of the cylinder?

336. A sphere can be circumscribed about any given tetrahedron.

337. Define: dihedral angle, projection of a line upon a plane, symmetric trihedral angles, prism, right cylinder, sphere.

338. The volume of a sphere is $2929\frac{1}{3} \pi$ cu. in.; find its surface.

339. Find the ratio of the volumes of two similar tetrahedrons whose homologous edges are as 1:8. Find the ratio of their edges if their volumes are as 1:8.

340. Find the locus of a point on a sphere that is equidistant from two given points.

341. A sphere of radius 10 in. is cut into two parts by a plane whose distance from the center of the sphere is 6 in. Find correct to three significant figures (a) the volume of the sphere, (b) the volume of the smaller segment.

342. The area of the base of a cone of revolution is b , and the area of the curved surface is c . Compute the volume of the cone.

✓ 343. Prove that two similar tetrahedrons are to each other as the cubes of any two homologous edges.

344. The area of the base of a right prism is 12 sq. in.; its total area is 295 sq. in.; the base is a regular hexagon. Find its volume.

345. Find the ratio of the edge of a cube to the edge of a regular tetrahedron having the same volume.

346. If the area of a spherical surface is 100 sq. ft., what is the area of a spherical triangle whose angles are 30° , 120° , and 150° ?

347. A cone of revolution and a cylinder of revolution each have as base a great circle of radius r , and as altitude the radius of the sphere. Find the ratios of the total surfaces of the cone and cylinder to the surface of the sphere.

348. If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other.

349. The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms. If the parallelepiped had been a right parallelepiped, what simpler method of proof could have been used?

350. The intersection of two planes tangent to a circular cylinder is parallel to the elements of the cylinder.

351. The inside of a glass is in the form of a cone whose vertical angle is 60° , and whose base is 2 in. across. The glass is filled with water, and the largest sphere that can be immersed is placed in the glass. How much water remains in the glass?

✓ 352. Find the altitude of a frustum of a circular cone, if its volume equals 190 cu. cm., and the radii of its bases are respectively 2 cm. and 3 cm.

353. Find the volume of a pyramid whose base contains 30 sq. in., if one lateral edge is 5 in., and the angle formed by this edge and the plane of the base is 45° .

354. A solid metal sphere whose radius is 12 in. is recast into a spheric shell; the cavity is spheric and has the same radius as that of the original sphere. Find the thickness of the shell.

355. Prove that any section of a tetrahedron made by a plane parallel to two opposite edges is a parallelogram.

356. Prove that the volume of any regular pyramid is equal to one third its lateral area multiplied by the perpendicular distance from the center of the base to any lateral face.

357. Prove that if the four sides of a spheric quadrilateral are equal, its diagonals bisect each other.

358. What is the locus of a point in space equally distant from three points? Demonstrate.

359. What is the locus of a point which is equidistant from three planes which meet in three parallel lines? Prove your answer.

360. If the four diagonals of a prism, whose base is a quadrilateral, meet in a point, prove that the prism is a parallelepiped.

361. The corner of a cube is cut off by a plane passing through the midpoints of the edges which terminate at that vertex, and the process is repeated for each corner of the cube. Prove that the volume of the solid that remains is five sixths of the volume of the cube.

362. Find the locus of a point whose distance from a fixed straight line is equal to its distance from a fixed plane perpendicular to that line.

363. Show that the area of a spherical triangle drawn on a sphere whose radius is 10 in., is $(200\pi - 270)$ sq. in., if the lengths of the sides of its polar triangle are respectively 8, 9, and 10 in.

364. Find the locus of a point equidistant from the three faces of a trihedral angle. Give proof.

365. Find the area and the volume of a solid generated by the revolution of an equilateral triangle about one of its sides as an axis, each side of the triangle being 4 in.

366. A rectangle 6 ft. by 4 ft. is revolved about each of two of its adjacent sides. Find the volume and the total surface of each solid generated.

367. The total surface of a right cylinder is S and the altitude is equal to the diameter of the base. Find, in terms of S , the volume of the cylinder.

368. A regular hexagonal pyramid whose altitude is 8 inches, and whose base is $54\sqrt{3}$ sq. in., is cut by a plane parallel to the base; the area of the section is $\frac{1}{4}$ the area of the base. Find (a) the distance of the section from the base; (b) a side of the section.

369. An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.

370. Find the surface of a lune whose angle is 30° , on a sphere whose radius is 5 ft.

371. If the altitude of a cone of revolution is equal to the diameter of its base, find the altitude if the volume equals that of a given sphere whose diameter is d .

372. An equilateral triangle is revolved about one of its altitudes as an axis. If a side of the triangle is 18, find the volume of the cone thus generated, and the area of the surface of the sphere inscribed in it.

373. Find the volume of a regular hexagonal pyramid the sides of whose bases are 16 and 8 respectively, and whose lateral edge is $8\sqrt{10}$.

374. (a) The area of a zone is 30π , and its altitude is 5. Find the radius of the sphere.

(b) On the preceding sphere, the sides of a triangle are 70° , 80° , 90° . Find the area of the polar triangle.

375. If from any point within a regular tetrahedron perpendiculars are dropped to the four faces, the sum of these perpendiculars is equal to the altitude of the pyramid.

376. Find the volume of a regular tetrahedron each of whose edges is 6 in.

377. The altitude of the frustum of a pyramid is 7.2 in., the lower base is 10 in. square, and the upper base 4 in. square. Find the volume of the frustum.

378. Find the volume and the convex surface of a cone of revolution circumscribing a regular hexagonal pyramid each edge of whose base is 8 in., and whose altitude is 15 in.

379. The base of a right prism 50 cm. high is a rhombus, whose longer diagonal is 30 cm., and whose shorter diagonal is 16 cm. Find the total surface of the prism.

380. A leaden cylinder of revolution 10 cm. long and 4 cm. in diameter is melted and cast into a hemisphere. Find the radius of this sphere.

381. The center of each of two spheres whose common radius is 4 in. is at the surface of the other. Find the volume of the solid common to both spheres.

382. Find the locus of points equidistant from two intersecting planes, and at the same time equidistant from two points.

383. If two spherical triangles on the same sphere are mutually equilateral, their polar triangles are mutually equiangular.

384. The bases of a right prism 5 in. in height are isosceles triangles. The base of each triangle is 3 in. and the altitude is 2 in. Through the base of each isosceles triangle a plane is passed, intercepting the opposite lateral edge of the prism 1 in. from the same base of the prism. Find to one place of decimals the volume and the total surface of the solid cut from the prism by these two planes.

385. If two parallel planes are cut by a third plane, the alternate dihedral angles are equal.

386. The outside of a glass ink well is in the form of the frustum of a regular hexagonal pyramid, 6 cm. high, 5 cm. on a side of the bottom, and 4 cm. on a side at the top. The inside surface is in the form of a right circular cylinder, of radius 2 cm., terminated by a hemispherical surface of the same radius, the total depth being 4 cm. How many cubic centimeters of glass are there in the well?

387. A point moves so as always to be at the constant distance x from a fixed point A , and at a constant distance y from a fixed plane MN . Determine its locus, discussing the different cases which may occur.

388. Prove that a triangular truncated prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the vertices of the inclined section.

389. Find the volume of a truncated triangular prism, if its base contains 40 sq. cm., the lateral edges are respectively 5, 10, 20 cm., and the projection of the edge of length 5 cm. upon the base equals 4 cm.

390. Find the area of a spherical triangle if the perimeter of its polar triangle equals 297° , and the radius of the sphere equals 10 cm.

391. The radius of a base of a right circular cylinder equals r , the altitude of the cylinder equals h . Find the radius and the volume of a sphere whose surface is equivalent to the lateral surface of the cylinder. Construct the radius of the sphere if r and h are two given lines.

✓ 392. If the solid angle formed at the vertex of a triangular pyramid is a trirectangular one, and a , b , and c denote respectively the area of the lateral faces, and d the area of the base, prove that $a^2 + b^2 + c^2 = d^2$.

393. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

394. If a plane is passed through one of the diagonals of a parallelogram, the perpendiculars to the plane from the extremities of the other diagonal are equal.

395. A point moves so that, if any secant be drawn from it to a fixed sphere, the product of the whole secant and its external segment is constant. What is its locus?

396. If h is the altitude of a segment of one base in a sphere of radius r , the volume of the segment is equal to $\pi h^2 \left(r - \frac{h}{3} \right)$.

397. How far from the surface of a sphere of radius 2 ft. must a light be placed to illuminate one third of its entire surface?

398. Find the volume of the portion of a sphere, lying outside of an inscribed right circular cylinder, whose altitude is h and radius r .

399. A regular hexagonal pyramid whose base is inscribed in a circle of radius 3 in. has an altitude of 4 in. Find the dimensions of a similar pyramid such that its lateral area shall be equal to the lateral area of the first increased by the surface of a sphere of radius 4 in. Answer need not be simplified.

400. A solid right circular cone of altitude 3 in., radius of base 2 in., rests base downwards in an inverted hollow cone, similar to the solid cone, the axes of both cones being vertical and the joint between the two water-tight. If the vertex of the solid cone just reaches the level of the rim of the hollow cone, find how many cubic inches of water can be contained in the vessel so formed. Answer need not be simplified.

401. State and prove the theorems true of the figure formed by cutting through a pyramid by a plane parallel to the base.

402. What is the ratio of the volume of a cube to the volume of a second cube whose edge is a diagonal of the first cube?

403. Find the locus of the center of a sphere which is tangent to three given planes.

404. How many rubber balls 2 in. in diameter and $\frac{3}{16}$ in. thick can be made from a flat circular sheet of rubber 1 in. thick and 2 ft. in diameter?

405. Find the portion of the earth's surface bounded by a geodetic triangle whose sides are 60° , 90° , and 100° respectively.

406. Name and describe the regular polyhedrons.

407. What portion of the surface of the earth is included between the parallels of 30° north and south latitude?

408. State (without proof) the theorem for the area of any spherical triangle, and illustrate by computing the area of an equilateral spherical triangle, each of whose angles is 70° , on a sphere of radius 5 feet.

409. What kind of figure is any section of a cone made by a plane passing through the vertex and cutting the base? by a plane parallel to the base? Prove your answers.

410. If the altitude of a cylinder of revolution equals the diameter d of a given sphere, determine the radius of the base of the cylinder so that (a) the total surfaces shall be equal, (b) the volumes shall be equal.

411. Find the volume of a spherical shell bounded by two concentric spheres whose surfaces are 20π and 15π .

412. Enumerate the various cases of equal spherical triangles, and prove any one of them.

413. In a right triangle whose sides are a and b , a line c is drawn bisecting a and parallel to b . Find the ratio of the surfaces generated by c and the hypotenuse of the original triangle when the figure is revolved about b .

414. Two truncated prisms are equal if three faces including a trihedral angle of one are respectively equal to three faces similarly including a trihedral angle of the other.

415. If an equilateral triangle and its inscribed circle are revolved about an altitude of the triangle, then the ratio of the volumes of the cone and the sphere thus generated are as 9 to 4.

416. What is the shortest line which can be drawn on the surface of a sphere between two points?

417. To determine a point in a given straight line which shall be equidistant from two given points in space.

418. If a plane is passed through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

419. A cylinder of revolution is inscribed in a sphere of radius 6 in. The altitude of the cylinder is twice the radius of the base. Find its total surface.

420. To determine a point in a given plane which shall be equidistant from three given points in space.

421. A projectile has the shape of a cylinder of revolution surmounted by a conical cap. Its total length is 24 in., and the cylindrical part is three times as long as the conical end. The greatest diameter is 10 in. Find the volume and the total surface of the projectile.

422. A plane is passed through a sphere, bisecting at right angles one of the radii. Compare the areas of the two portions into which the spherical surface is divided.

423. To construct a sphere passing through two given points in space, and having its center in a given line.

424. Given a sphere 10 in. in diameter. A regular square pyramid is inscribed and its altitude is 8 in. Find the volume and total surface of the pyramid.

425. If the diameter of a sphere is doubled, in what ratios are the surface and the volume increased?

426. Given two intersecting planes in space, to determine the locus of the center of a sphere of given radius which shall be tangent to each of the given planes.

427. One and but one sphere can be inscribed in any tetrahedron,

428. In a trapezoid $ABCD$, AB and CD are parallel and AC is perpendicular to CD . The dimensions are, $AB = 4$ in., $CD = 12$ in., $AC = 6$ in. The figure is revolved about AC . Find the total surface and the volume of the conical frustum generated.

429. The diameter of a sphere of radius 5 in. is increased by 1 in. How much are the surface and the volume increased?

430. A point and any two non-intersecting straight lines are given in space. Show how to construct a straight line which shall pass through the given point and intersect each of the given lines.

431. An equilateral triangle and its inscribed circle are revolved about an altitude of the triangle, generating a cone of revolution and its inscribed sphere. Find the area of the circle along which the cone and sphere are tangent.

432. Find the area of a spherical triangle whose angles are 90° , 120° , 45° , on a sphere whose radius is 6 in.

433. Prove that if the three faces including a trihedral angle of a prism are respectively equal to the three faces including a trihedral angle of another prism, and are similarly placed, the two prisms are equal.

434. In a sphere whose radius is 13 is inscribed a cylinder whose altitude is 24. Find the ratio of the area of the surface of the cylinder to that of the sphere.

435. On a sphere whose radius is 12, find the altitude of a zone whose area is equal to the area of the spherical triangle whose angles are 120° , 110° , and 80° .

436. Prove that the four diagonals of a parallelepiped meet in a point, which is the midpoint of each.

437. Prove that every section of a parallelepiped made by a plane intersecting all its lateral edges is a parallelogram.

438. Construct a spheric surface with a radius r which shall pass through two given points and be tangent to a given plane. Prove your construction.

439. Let PQ be a line drawn from a given point P , meeting a given plane in Q . Find the locus of the point midway between P and Q when Q moves arbitrarily in the given plane. Give proof.

440. On a sphere of radius 10 in. find the area of a zone with altitude 3 in.

441. A right cone and a cylinder of revolution on the same base with radii 3 in. have the same altitude 4 in. Find the ratio of their volumes and of their lateral areas.

442. Find the capacity of a circular pail 15 in. high, the radius of the lower base being 4 in. and the radius of the upper base 6 in.

✓ 443. The spire of a church is a right pyramid on a regular hexagonal base, each side of the base is 10 ft., and the height is 50 ft. There is a hollow part which is also a right pyramid on a regular hexagonal base, the height of the hollow part being 45 ft., and each side of the base 9 ft. Find the number of cubic feet of stone in the spire.

444. The volumes of two similar polyhedrons are 64 cu. ft. and 216 cu. ft. respectively. If the area of the surface of the first is 112 sq. ft., what is the area of the surface of the second?

445. Find the volume of a cone of revolution the area of whose total surface is 200π sq. ft., and whose altitude is 16 ft.

446. Find the area of a spherical triangle whose angles are 100° , 120° , 140° , the diameter of the sphere being 16 in.

447. In any trihedral angle the common intersection of the three planes bisecting the three dihedral angles is a straight line.

448. To what extent is a straight line determined by the condition that it is perpendicular to (a) two intersecting straight lines, (b) two non-intersecting straight lines?

✓ 449. If a section parallel to the base of a pyramid is equal to one ninth of the base in area, the altitude of the pyramid being 6 ft., how far from the vertex is the plane of the section?

✓ 450. A solid is in the form of a right circular cone, standing on the flat base of a hemisphere of equal base, radius 10 inches; the volume of the cone is equal to that of the hemisphere. Determine the surface and the volume of the smallest right circular cylinder that will contain this solid, the axis of the cylinder being the same as that of the solid. How much waste space is there in the cylinder?

451. Explain why a circular table with three legs, symmetrically placed, will be perfectly steady if placed on a plane floor, even though one of the legs is slightly longer than the other two, while a table with four legs, of which one is a little longer than the others, will not be steady.

452. Two angles of a spherical triangle are 80° and 120° . Find within what limits the third angle must lie; and prove that the greatest possible area the triangle can have is four times the least possible area, the sphere on which it is drawn being given.

453. Define a regular polyhedron, and prove that there cannot be more than five regular polyhedrons. Assuming that there are five, state for each (a) the nature of the faces, (b) the number meeting at a vertex, and (c) the number of faces. How many vertices has each of the regular polyhedrons? how many edges?

454. Prove that any right circular cone can be produced by the revolution of a triangle about one side as an axis. What kind of triangle must be used?

455. An irregular portion (less than half) of a material sphere is given. Explain how the radius can be found, compass and ruler being allowed.

456. Give expressions for the surface and volume of a sphere of radius r and prove one of them.

457. A solid sphere of metal of radius 18 in. is recast into a hollow sphere. If the cavity is spherical, of the same radius as the original sphere, find the thickness of the shell.

458. The volumes of a hemisphere, right circular cone, and right circular cylinder, are equal. Their plane bases are also equal, each being a circle of radius 10 in. Find the height of each.

459. A sphere of radius 5 ft. and a right circular cone on a base also of radius 5 ft. stand on a plane. If the height of the cone is equal to the diameter of the sphere, find the position of the plane that cuts the two solids in equal circular sections. Find the area of these sections.

460. Find the area of a spherical triangle whose angles are 100° , 120° , 75° , if the radius of the sphere is 7 in.

461. The vertices of one regular tetrahedron are the centers of the faces of another regular tetrahedron. Find the ratio of the volumes.

462. Find the angle of an equiangular spherical hexagon equivalent to the sum of three equiangular spherical hexagons whose angles are equal to 130° .

✓463. The radii of two spheres are 13 ft. and 15 ft. respectively, and their line of centers equals 14 ft. Find the volume of the solid common to both spheres (spherical lens).

464. Find the altitude of a frustum of a circular cone, if its volume equals 190 cm., and the radii of the bases are respectively 2 cm. and 3 cm.

465. Assuming the earth to be a sphere with a diameter equal to 8000 mi., find the area of a zone bounded by the parallels of 30° north and 30° south latitude.

466. The lateral surface of a right circular cone is equal to three times the area of the base. If the radius of the base equals r , find the altitude and the volume of the cone.

467. Two opposite angles A and C of a spherical quadrilateral $ABCD$ are equal, and AB and CB are produced through B to meet the opposite sides in E and F respectively. If angle E equals angle F , prove that $AB = BC$. Is the corresponding proposition for plane geometry correct? Give reasons.

468. State a proposition that may be used to construct a plane parallel to a given line. State a proposition that may be used to construct a plane perpendicular to a given plane. Through a given point to construct a plane parallel to a given line and perpendicular to a given plane.

469. From the vertices of a triangle ABC perpendiculars AA' , BB' , and CC' are dropped upon a straight line XY in the same plane. $AA' = 2$, $BB' = 3$, $CC' = 1$, $A'B' = 2$, $B'C' = 1$, and $C'A' = 3$. Find the volume generated by the revolution of triangle ABC about XY as an axis.

470. Find the volume of a pyramid whose base contains 30 sq. cm. if one lateral edge is 5 cm., and the angle formed by the edge and the plane of the base is 45° .

✓471. Prove that two triangles on the same sphere that are mutually equiangular are mutually equilateral. Are such triangles necessarily equivalent? Are they necessarily equal?

✓472. If a spherical quadrilateral is inscribed in a small circle, prove that the sum of two opposite angles is equal to the sum of the other two angles.

✓ 473. On the base of a right circular cone a hemisphere is constructed so that it lies outside the cone, and the surface of the hemisphere equals the surface of the cone. If r is the radius of the hemisphere, find (a) the slant height of the cone, (b) the inclination of the slant height to the base, (c) the volume of the entire solid.

474. Find the total surface and the volume of a regular tetrahedron whose edge equals 8 cm.

475. Let AB and CD be two lines not in the same plane, and let E be any point on AB and F any point on CD . What is the locus of the middle point of the straight line EF ? Prove the truth of your answer.

476. The altitude of a cone of revolution is 12 cm. and the radius of its base is 5 cm. Compute the radius of the sector of paper which, when rolled up, will just cover the convex surface of the cone; and compute the size of the central angle of this sector in degrees, minutes, and seconds.

477. Regarding the earth as a sphere, show that $\frac{11}{8}$ of its volume is included between the planes of the small circles of 30° north latitude and 30° south latitude.

478. If the area of a zone of one base is n times the area of the circle which forms its base, prove that the altitude of the zone is $\frac{n-1}{n}$ times the diameter of the sphere.

479. A variable sphere is tangent to a fixed plane at a fixed point of the plane, and a plane is drawn tangent to the sphere and parallel to a second fixed plane. What is the locus of the point of contact?

↓ 480. Prove that the edge of a regular octahedron is approximately 2.45 times the radius of the inscribed sphere.

481. Prove that if the four sides of a spheric quadrilateral are equal, its diagonals bisect each other.

482. Define: (a) a perpendicular to a plane; (b) the distance from a point to a plane; (c) the angle that a line makes with a plane; (d) a right circular cylinder; (e) a spherical polygon.

483. Prove that if a plane and a line not in that plane are perpendicular to the same plane, they are parallel to each other.

484. If from a point P of a perpendicular PA to a plane MN a perpendicular PD is drawn to a line BC of the plane, any line DA that joins the foot D of the second perpendicular to any point A of the first perpendicular will itself be perpendicular to the line BC of the plane MN .

485. If two lines in space are non-intersecting, are their projections on a plane necessarily non-intersecting? Illustrate with a figure.

486. Is it true that if two lines are perpendicular to each other, any plane passed through one of the lines is perpendicular to the other? Give the reason for your answer.

487. Is it possible for two planes to be perpendicular to each other if both are perpendicular to a third plane? Illustrate your answer by a figure.

488. Prove that if two planes are parallel to a third they are parallel to each other. Is this theorem true if the word "parallel" is replaced by "perpendicular"?

489. Prove that if a straight line is perpendicular to a plane, every plane passed through that line is perpendicular to the plane. Is it true that, if a plane is perpendicular to a second plane, every line of the first plane is perpendicular to the second plane? Illustrate by a figure. Is it true that, if a plane is parallel to a second plane, every line of the first plane is parallel to the second plane? Give the reasons.

490. Prove that the lateral area of a prism is equal to the product of the perimeter of a right section of the prism and a lateral edge. Is this product the same as the product of the perimeter of the base and an altitude? Give the reason for your answer.

491. Show how to cut a tetrahedron by a plane so that the section shall be a parallelogram. In what three ways may the cutting plane be passed? When is it possible to get a rectangular section?

✓ 492. The vertices of a tetrahedron are four vertices of a cube, no two of which lie on the same edge of the cube. Prove that the volume of the tetrahedron is one third the volume of the cube.

493. Prove that if a pyramid is cut by a plane parallel to the base, the section is a polygon similar to the base. Where would such a section be placed if its area were one half that of the base?

494. If two non-coplanar lines are cut by three parallel planes, prove that the corresponding segments are proportional. If a triangular pyramid is cut by two parallel planes whose distances from the vertex are 2 ft. and 5 ft., what can you say about the area of the sections thus formed?

495. Assuming that the volume of a rectangular parallelepiped is equal to the product of its base by its altitude, state in logical order, without proof, the theorems that lead to the volume of any cylinder.

496. From a cylinder of revolution whose altitude is 12 in. and whose diameter is 10 in. is cut out a cone. The base of the cone coincides with one base of the cylinder, and its vertex is at the center of the other base of the cylinder. Compute the total surface and the volume of the solid that remains.

497. The area of a certain small circle on a sphere is equal to the difference of the areas of the zones into which the circle divides the spherical surface. Find the distance from the center of the sphere to the plane of the small circle, if the radius of the circle is r . If a cone is tangent to the sphere along the given circle, show that the distance from the center of the sphere to the vertex of the cone is $(\sqrt{5} \times 2)r$.

498. From a hemisphere whose radius is 3 in. a solid is cut by means of a cone of revolution whose base coincides with the base of the hemisphere, and whose altitude equals the radius of the sphere. Find the volume and the surface of the solid that is left, correct to two decimal places.

499. A hemispherical dome 140 ft. in diameter is divided into two parts by a horizontal plane that lies seven eighths of the way from its base to the summit. What will be the expense of gilding the lower part at a cost of 25 cents per square foot?

500. Show that if a man ascended in a balloon to a height equal to the earth's radius, he would see one quarter of the earth's surface.

501. Find the volume of a spherical segment of one base whose altitude is one half the radius of the sphere.

502. Derive the formula for the volume of a solid included between a spherical surface of radius r , and a conical surface tangent to the sphere, the vertex of the cone being distant $2r$ from the center of the sphere.

503. A certain dome is in the form of a spherical zone of one base. The base is a circle 24 ft. in diameter, and the highest point is 9 ft. above the plane of the base. Find the number of square feet in the surface of the dome, correct to two decimal places.

504. What proportion of the earth's surface lies between the equator and latitude 45° ? Draw the figure.

505. Find the area of a spherical quadrilateral whose angles are 135° , 90° , 80° , and 70° , on a sphere of radius 10 ft.

506. The dihedral angles of a trihedral angle are 100° , 65° , 87° . The trihedral angle is closed by a portion of a spherical surface whose radius is 4 in., and whose center is at the vertex of the trihedral angle. Compute to two decimal places the area of this portion of the spherical surface.

507. Prove that if two arcs of great circles intersect on the surface of a hemisphere, the sum of the opposite spherical triangles which they form is equivalent to a lune whose angle is the angle between the arcs in question. In what connection is this proposition important?

508. Assuming the area of a triangle as known, give, without detailed proof, the logical steps that lead to the formula for the lateral area of a right circular cone. Does this formula apply to an oblique cone? Give your reasons.

509. If a rectangle revolves about its shorter side a , and then about its longer side b , which of the cylinders thus generated will have the greater volume?

510. Assuming the volume of a cube as known, give, without detailed proofs, the logical steps which lead to the formula for the volume of a right circular cylinder. Does this formula apply also to an oblique circular cylinder? Give your reasons.

511. The chimney of a factory has the shape of a frustum of a regular pyramid. It is 180 ft. high, and its upper and lower bases are squares whose sides are 10 ft. and 16 ft. respectively. The flue is throughout a square whose side is 7 ft. How many cubic feet of material does the chimney contain?

512. The corner of a cube is cut off by a plane passed through the outer extremities of the three edges meeting at the given corner. What part of the volume of the cube is thus removed?

513. How far from the vertex of a pyramid must a section parallel to the base be passed in order that the product of this section by the altitude of the pyramid shall equal the volume of the pyramid?

514. The volumes of two given spheres are in the ratio 1:2. Find the ratio of the total surfaces of their inscribed cubes.

515. Show how to construct a regular tetrahedron.

516. A right triangle whose legs are 3 and 4 revolves about the longer side. Show that the total surface thus generated is less than the surface of a sphere whose diameter equals the hypotenuse of the given triangle.

517. State and prove the theorems true of the figure formed by cutting through a pyramid by a plane parallel to the base.

518. In a sphere whose diameter is 14 in. the altitude of a zone of one base is 2 in. Find the altitude of a cylinder of revolution whose base is the base of the zone, and whose lateral area equals the area of the zone.

519. Given two points A and B in two intersecting planes M and N . To find Z in the line of intersection of M and N such that $AZ + XB$ shall be a minimum.

✓✓ 520. What are the ratios of the radii and the volumes of two spheres, one having 81 times the surface of the other? What are the ratios of the areas and radii of two spheres, one having 81 times the volume of the other.

521. Two planes perpendicular to the same plane P , and containing two lines AB and $A'B'$ parallel to each other, are parallel. Show also that this is not true if the lines AB and $A'B'$ are perpendicular to plane P .

522. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases.

523. Show that the projections of parallel lines on the same plane are parallel, but that the converse is not true.

524. The area of the surface generated by the base AB of the isosceles triangle OAB , which revolves about a fixed axis XY lying in its plane and passing through its vertex without cutting the triangle, is equal to the circumference which has for its radius the altitude OT of the triangle, multiplied by the projection of the base AB on the axis XY .

525. If, from any point on a sphere as a pole, with a polar distance equal to one third of a quadrant, we describe a circle on the sphere, the radius of this circle will be half the radius of the sphere.

526. If M and N are the feet of the normals to two planes from a point P , show that the line of intersection of the two planes is normal to the plane MNP .

527. The base of a prism of altitude 7 in. is a regular hexagon whose side is 4 in. The edges of the prism make angles of 60° with the altitude. Find the lateral area and the volume of the prism.

528. Through a point A are drawn three mutually perpendicular lines, AB , AC , and AD , B , C , and D being any points on the three lines. The points C and D are joined, and from A a perpendicular is dropped to this line CD , meeting it in E . Prove that the line joining E to any point in AB is perpendicular to CD .

529. The three lines joining the middle points of the three pairs of opposite edges of a triangular pyramid meet in a point.

530. The area of the entire surface of a frustum of a right circular cone is 120π , and the diameters of the parallel bases are 8 and 14. Find the lateral area and the volume of the frustum.

531. The surfaces of a sphere, the circumscribed right cylinder, and the circumscribed right cone whose axial section is an equilateral triangle are in the ratio 4 : 6 : 9. The same is true of the volumes.

532. In grading the site for a summer house, a mound is to be made in the shape of a truncated right circular cone. The lower base has a diameter of 12 meters, the upper base (4 meters higher), of 8 meters. How many cubic meters of earth will be necessary?

533. If two lines intersect at right angles, under what conditions will their projections upon a given plane be perpendicular?

534. Find the volume of the frustum of an oblique pyramid from the following data: the lower base is a square of side 4 in.; the upper base a square of side 2 in.; one of the inclined edges, which is 8 in. long, has as its projection upon the lower base one of the diagonals of that base.

535. Find the length of the projection upon a plane of a line 8 in. long (a) parallel to the plane, (b) making an angle of 60° with the plane, (c) perpendicular to the plane.

536. Find the number of cubic feet of concrete in a dam 250 ft. long, 31 ft. high, 33 ft. wide at the bottom, and 5 ft. wide at the top.

537. Find the volume of a regular tetrahedron whose slant height is $\sqrt{3}$.

538. Find the length of wire $\frac{1}{16}$ in. in diameter that can be made from a cubic foot of copper.

539. The volume of a regular square pyramid is $43\frac{1}{2}$ cubic ft.; its altitude is twice one side of the base. Find (a) the lateral area of the pyramid, (b) the area of a section made by a plane parallel to the base and 1 ft. from the base.

540. In a triangle, the sides including an angle of 120° are respectively a and $2a$. Find the volume of the solid generated by revolving the triangle about the shorter side as an axis.

541. A solid glass ball 6 in. in diameter is expanded by a glass blower till the glass is 1 in. thick. Find the outer diameter of the hollow globe.

542. The altitude of a cone, the diameter of its base, and the edge of a given cube are equal. Find the ratio of the volume of the cone to the volume of the cube.

543. The sides of a parallelogram which are 12 in. and 8 in. respectively form an angle of 60° . Find the volume and the convex surface of the solid generated by the revolution of the parallelogram about one of its longer sides as an axis.

544. Three plane angles are 20° , 80° , 105° . Will these angles form a trihedral angle? Why?

545. A cylindrical water pipe has a diameter of a inches and a velocity of flow of b ft. per second. How many cubic feet of water will be discharged in one minute at a given point, no allowance being made for friction? Also, what would be the effect upon the quantity if the diameter of the pipe were doubled?

546. A circular sector whose angle is 60° , and whose radius is 6, revolves about a diameter perpendicular to one of the limiting radii. Find the volume of the spherical sector generated.

547. The volumes of two similar cones of revolution are to each other as 343 is to 512. What is the ratio of the lateral surfaces?

548. Define symmetry as to a center, and show that the symmetrical figure of a dihedral angle is an equal dihedral angle.

549. The total surface of a regular tetrahedron is 60 sq. m. Find its volume.

550. Find the locus of those points of a plane at which a given straight line, not lying in that plane, subtends a right angle. Show when the locus becomes a point and disappears.

551. The volume of a sphere inscribed in a regular tetrahedron is 2 cu. in. What is the volume of a circumscribed sphere?

552. Show how to cut a given polyhedral angle of four faces so that the section shall be a parallelogram.

553. What part of the surface of a sphere is illumined by a lamp placed at a distance of a diameter from the surface of the sphere?

✓ 554. Two tetrahedrons which have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of the equal trihedral angles.

555. A right circular cone the altitude of which is three times the radius of its base, and a sphere, the radius of which is equal to the radius of the base of the cone, are immersed in a rectangular cistern of water the base of which is 9 ft. by 11 ft. If they are removed, the water level is lowered by 2 ft. Find their dimensions.

556. A cylindrical tank 10 ft. long and 5 ft. in diameter lies with its axis horizontal. If it contains gasoline to a depth of 15 in., how many gallons are in the tank? (231 cu. in. to the gallon.)

557. Find the point the great circle distances of which from the sides of a spherical triangle are equal. State your construction.

558. Four of the six planes determined by the diagonals of a parallelepiped divide the parallelepiped into six quadrangular pyramids. Prove that these six pyramids are equivalent.

559. Describe a spherical surface with a given radius that shall pass through two given points and be tangent to a given plane.

560. The axis of a right cylinder passes through the centers of all sections parallel to the base.

561. The base of a pyramid is a right triangle whose base is 12 in. and whose hypotenuse is 20 in. The altitude of the pyramid is 15 in. Find the volume of the frustum of this pyramid cut off by a plane 3 in. above the base.

562. The base of a right prism is a rhombus, one side of which is 10 in., and the shorter diagonal is 12 in. The altitude is 15 in. Find the volume.

563. A regular hexagonal pyramid is cut into two parts of equal volume by a plane parallel to the base. What is the distance from the vertex to this plane, in terms of the altitude?

564. Find the volume of a right pyramid whose slant height is 13 ft., and whose base is an equilateral triangle inscribed in a circle whose radius is 10 ft.

565. The volume of the frustum of a square pyramid is 74 cu. in.; the edges of the bases are 3 in. and 4 in. respectively. Find the altitude.

566. The altitude of a regular pyramid is $2a$, and the base is a triangle inscribed in a circle of radius a . Find the lateral area of the pyramid.

567. Find the area of a zone on a sphere of radius r that is illumined by a lamp placed at a distance a from the surface.

568. A cone of wood has its vertex angle equal to 60° , and the radius of its base equal to 2 in. A cylindrical hole of radius 1 in. is bored through the entire cone, the axes of the two coinciding. How much of the cone goes into chips?

569. The distance between two parallel planes is 16 in. A line 24 in. long has an extremity in each of these planes. Find the length of the segments into which this line is divided by a plane parallel to the given planes and 4 in. from one of them.

570. What is the greatest number of faces that a convex polyhedral angle can have if each face angle is 60° ? Why?

571. If the area of a section is one third that of the base, what is the ratio of the segments into which the altitude is divided?

572. Define polar triangle of a spherical triangle. If a is the side of a spherical triangle, and A' the opposite angle of the polar triangle, prove that $A' + a = 180^\circ$.

573. From a point 6 ft. from the surface of a sphere, one quarter of its surface is visible. Find the radius of the sphere.

574. A triangle whose sides are respectively 15 in., 13 in., and 4 in., revolves about the shortest side as an axis. Find the volume of the solid generated by the revolving triangle.

575. The volume of a regular hexagonal prism is $81\sqrt{3}$; the altitude of the prism is equal to the longest diagonal of the base. Find the total area of the prism.

576. Find the volume of a hemisphere whose entire surface equals S .

577. Find the locus of a point on a sphere that is equidistant from two given points on the surface.

578. The volume of a sphere is 4500π cu. in. Find its surface.

579. Find the surface and the volume of the solid generated by a door 3 ft. wide and 8 ft. high swinging in an arc of 144° .

580. The hypotenuse of a right triangle is 5 in., one of its legs is 3 in. Find the volume of the solid generated by revolving the triangle on its hypotenuse as an axis.

581. Assuming that the radius of the earth is 4000 mi. and that the crust is 30 mi. thick, find the volume of the crust of the earth.

582. Prove that if a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

583. Find the weight of 52,800 linear feet of copper wire $\frac{5}{8}$ of an inch in diameter. (1 cu. ft. of copper weighs 556 lb.)

584. Find the number of cubic feet of earth in a railway embankment 2500 ft. long, 10 ft. high, 12 ft. wide at the top, and 42 ft. wide at the bottom.

585. Find the cost, at \$2.50 a square foot, of gilding a hemispheric dome whose diameter is 50 ft.

586. A sphere of lead 10 in. in diameter is melted and cast into a cone 10 in. high. Find the diameter of the base of the cone.

587. Find the capacity in cubic inches of a berry box in the form of a frustum of a pyramid 5 in. square at the top, and $4\frac{1}{2}$ in. square at the bottom, and $2\frac{3}{4}$ in. deep.

588. Two tanks are in form similar solids; one holds 128 gal., the other 250 gal. If the first is 20 in. deep, find the depth of the second.

589. The total surface of a cube is 450 sq. in. Find the volume.

590. Define as loci (1) the intersection of two planes; (2) the bisecting plane of a dihedral angle; (3) the plane which is the perpendicular bisector of a given sect; (4) the plane perpendicular to a given line at a given point.

591. Define (1) the angle between a line and a plane; (2) a parallelepiped; (3) symmetric polyhedral angles; (4) the volume of a geometric solid.

592. Find the number of cubic yards of dirt to be excavated in digging a canal, 50 ft. wide at the top, 30 ft. wide at the bottom, 14 ft. deep on the average, between two locks 2.6 mi. apart.

593. Compute the volume of a regular tetrahedron whose slant height is $\sqrt{3}$.

594. The sides of a parallelogram, which are 12 in. and 8 in. respectively, form an angle of 60° . Find the volume and the convex surface of the solid generated by the revolution of the parallelogram about one of its longest sides as an axis.

595. Prove that the smallest section of a sphere made by a plane passing through a given point within the sphere is that made by a plane perpendicular to the radius through the given point.

596. The lateral area of a right cylinder is 48π ; the volume is 96π . Find the radius and the height of the cylinder.

597. An isosceles trapezoid revolves about its longer base as an axis; the bases are respectively 14 in. and 8 in., the legs each 5 in. Find the surface of the solid generated.

598. The radius of a sphere is 20 in. Find the area of a section made by a plane 5.6 in. from the center of the sphere.

599. Find all possible locations of a point that is equidistant from two given points in space and at a given distance from a third point.

600. A cone 5 ft. high is cut by a plane parallel to the base and 2 ft. from the base; the volume of the frustum formed is 294 cu. ft. Find (a) the volume of the cone, (b) the volume of the part cut off by the plane.

601. A cylindrical tank 20 ft. long and 9 ft. in diameter lies with its axis horizontal. If it contains gasoline to a depth of 6 ft., how many gallons are in the tank? (231 cu. in. to the gallon.)

APPENDIX

318. Statements whose Conditions and Conclusions may have Two or More Parts. If the condition or the conclusion of a statement is composed of two or more parts, the consideration of its related statements can be made very complicated by taking those parts in all their different combinations as condition and conclusion. The full discussion of such possibilities has no place in Geometry, but belongs in a course on Logic. The following illustration will, however, show how the different combinations can be used to find new propositions. It will be considered only in reference to the converses.

In an isosceles triangle, a line through the vertex parallel to the base bisects the exterior angles at the vertex.

CONDITIONS :

The triangle is isosceles,
The line passes through the vertex,
The line is parallel to the base.

CONCLUSION :

The line bisects the exterior angles at the vertex.
(Since the bisector of one angle also bisects the other, this is considered as one part.)

COMPLETE CONVERSE : The interchange of the entire condition with the entire conclusion would give

If a line bisects the exterior angles at the vertex of a triangle, (1) the triangle is isosceles; (2) the line passes through the vertex; (3) the line is parallel to the base.

Of these three conclusions, (2) obviously is true ; (1) and (3) are not true from the condition alone, for the bisector of the angle cannot affect the shape of the triangle, neither is a bisector always parallel to the base of a triangle, no matter how the triangle is drawn. If, however, either (1) or (2) is taken as part of the condition, and the other is taken as the conclusion, a new proposition results, as,

In an isosceles triangle, the bisector of the exterior angle at the vertex is parallel to the base. Or,

If the bisector of an exterior angle of a triangle is parallel to the base, the triangle is isosceles.

In applying logic to elementary geometry, it is seldom necessary to consider the general description of the figure as a part of the condition. For example, in

If two sides of a triangle are equal, the opposite angles are equal. The fact that a triangle is being considered is regarded as a general description, and the statement is therefore one having a single condition, and its converse is

If two angles of a triangle are equal, the opposite sides are equal.

319. Cavalieri's Theorem. — *If two solids contained between the same two parallel planes are such that their sections by any third plane parallel to those two planes are equivalent, the two solids have the same volume.*

Divide the common altitude of the two solids into n equal parts, and through the points of division pass planes parallel to the base planes. Let M and N be two consecutive parallel planes, and let M cut the solids in the equivalent sections s_1 and s_2 . At some point in the perimeter of s_1 erect a line between M and N perpendicular to M , and let it generate a cylindrical or prismatic surface, or a combination of them, by moving always parallel to its original position, with s_1 as a guiding line. Do the same with s_2 , thus generating two surfaces which include right solids between the planes M and N . These solids have equivalent bases and the same altitude, and are therefore equivalent in volume.

If the same operation is performed with each consecutive pair of parallel planes, two sets of right solids will be formed, such that each corresponding pair are equivalent in volume, and their sums are also equivalent. Their sums differ somewhat from the two given solids, but if the number of divisions of the altitudes is increased without limit, the sums of the right figures will approach the given solids as limits. But, since the sums are equal variables, their limits are also equal, and the given solids are equivalent in volume.

Among the various sets of Cavalieri bodies are the following: prisms, cylinders, and combinations of prisms and cylinders, of equivalent bases and equal altitudes; pyramids, cones, and combinations of pyramids and cones, of equivalent bases and equal altitudes; a sphere of radius r , a right cylinder of radius r and altitude $2r$, hollowed out in conical form from each base to the center, and a tetrahedron with two of its edges in two planes tangent to the sphere at the ends of a diameter, and with its midsection parallel to those edges and equivalent to a great circle of the sphere.

320. The Prismatoid. The statement of the formula for the volume of a prismatoid, and the figures of elementary geometry to which it applies, were given in § 182. It is now required to prove the formula for the prismatoid itself.

Given the prismatoid of bases b_1 and b_2 , midsection m , and altitude h ; to prove $V = \frac{h}{6}(b_1 + b_2 + 4m)$.

Draw a diagonal in each quadrilateral face (as KL), thus dividing the entire lateral surface into triangles.

Take P , any point in m , and by passing planes through P and each lateral edge of the figure, including the diagonals drawn in the faces, form pyramids from P as a vertex to each face, or triangle formed in a face, as a base. Then the prismatoid is divided into pyramids of which it is the sum; that is, $P-b_1 + P-b_2 + P-RST +$ pyramids from P to each other triangular lateral face = the prismatoid.

On considering any one of the pyramids from P to a lateral face, as $P-RST$, where PXY is the part of m cut off by the faces of the pyramid:

$P-RST = 4(P-XYT)$, because $RST = 4XYT$. (Why?) But $P-XYT$, with T as the vertex,

$$= \frac{h}{6} PXY, \text{ so } P-RST = \frac{4h}{6} PXY.$$

Since each pyramid from P to a lateral triangle equals $\frac{4h}{6}$ times its part of m , their sum equals $\frac{4h}{6}m$.

But pyramid $P-b_1 = \frac{h}{6} b_1$, and pyramid $P-b_2 = \frac{h}{6} b_2$,

therefore $V = \frac{h}{6} (b_1 + b_2 + 4 m)$.

Besides this formula for the prismatoid, there is another formula that uses but one base, and instead of the mid-section, uses the section parallel to the bases, and two thirds the distance from the base used in the formula. If the section at the two-thirds point of the altitude is called s , this formula is $\frac{h}{4} (b_1 + 3 s)$.¹

321. Circular Cylindrical and Conical Surfaces. A cylindrical surface whose right section is a circle is a *circular cylindrical surface*. The surface of an oblique circular cylinder is not circular; and conversely, an oblique cylinder cut from the space inclosed by a circular cylindrical surface is not a circular cylinder. The two ideas, — that of the surface being circular, and that of the solid being circular, — are therefore distinct.

The formula for the area of a right circular cylinder can be extended to any cylinder by making the formula the element times the perimeter of a right section. It will not, however, use the circle in this case.

Similarly, a conical surface is circular if a section perpendicular to its axis of symmetry is a circle. The surface of an oblique circular cone is not circular, and an oblique cone cut from the space bounded by a circular surface is not a circular cone.

The area formula does not hold for the general case of the cone.

The volume formulas for the circular cylinder and the circular cone can be extended to all cylinders and cones.

¹ From *Metrical Geometry*, by Dr. George Bruce Halsted.

The formula for the cylinder is bh , and for the cone is $\frac{bh}{3}$, where b is the area of the base and h is the altitude. If a cylinder or a cone that is cut from a space whose surface is circular, has not circular bases, its bases are ellipses, and the area can be obtained by the formula πab , where a and b are the two half diameters (or axes) of the ellipse.

322. Comparison between Plane Geometry and Spherical Geometry. For many of the plane geometry propositions concerning rectilinear figures corresponding propositions have been proved in spherical geometry concerning great circle arcs. Corresponding propositions do not exist for all such plane geometry propositions because two straight lines must either be parallel or intersect in one point, while two great circle circumferences must meet in two points.

This difference eliminates those theorems which depend on parallels, including the propositions about similar figures, and those which depend upon the fact that two non-parallel lines can intersect in but one point. That it does not necessarily eliminate all those which use the single intersection is shown by the fact that two spherical triangles are congruent if they have two angles and the included side of one equal to the corresponding parts of the other, although in plane geometry the proof uses the fact that two lines meet in but one point. In the spherical triangles the proof still holds because the two great arcs can meet in but one point *on the hemispherical surface*, and in superposing the triangles, but one of the intersection points can therefore be used.

One of the groups of theorems most affected by the two

intersections is the one concerned with the sum of the angles of a triangle or other polygon. The foundation proof of this group is the proposition.

An exterior angle of a triangle is greater than either interior angle not adjacent to it.

That this does not hold in spherical geometry can be shown as follows:

In plane geometry, the line AR is drawn from A through M , the midpoint of BC , to R , so that $MR = AM$; and BR is drawn. Then $\triangle MBR \cong \triangle AMC$, and $\angle MBR = \angle C$. But since AR has met both arms of $\angle CBX$, it cannot meet either again, and R and BR are within that angle, and so $\angle MBR$ is part of, and is smaller than, $\angle CBX$, which proves the exterior angle greater than $\angle C$.

But, in spherical geometry, while all the rest of the proof holds, it is not true that AM extended cannot meet BX again, for it must meet it at a distance from A equal to a semicircumference. Therefore

1. If $AM < \text{a quadrant}$, R is within $\angle CBX$, as in position R_1 , and $\angle C < \angle CBX$.

2. If $AM = \text{a quadrant}$, R is on AX , as in position R_2 , and $\angle C = \angle CBX$.

3. If $AM > \text{a quadrant}$, R is outside $\angle CBX$, as in position R_3 , and $\angle C > \angle CBX$.

To show that *the exterior angle of a spherical triangle is always less than the sum of the non-adjacent interior angles, and therefore the sum of the angles of a spherical triangle is greater than a straight angle:*

If the exterior angle in the last figure is equal to, or less than, the interior angle, no farther proof is needed ; if it is greater than the one interior angle, that is, if AM is less than a quadrant, let the three great circumferences AC , AM , and AB meet again at A' , and continue BR to K on ACA' .

$\angle CAM = \angle MRB$ (cor. pts. of $\cong \triangle$) and $= \angle KA'R$ ($\angle$ of a lune).

$\angle MRB = \angle KRA'$ (vertical $\angle$ s); $\therefore \angle KRA' = \angle KA'R$,
and $KA' = KR$.

$\therefore KA' < KB$, and $\angle KBA' < \angle A'$ or its equal $\angle A$, and $\angle A + \angle B + \angle C > \text{st. } \angle$.

This is also shown by the use of polar triangles in § 277.

323. The General Polyhedron. Theorem. *In a polyhedron of f faces, e edges, and v vertices,*

$$e + 2 = f + v.$$

If the numbers of vertices and edges of any polyhedron are counted, beginning with any face, and then taking a second face, a third face, and so forth, always choosing a face that has one or more edges in common with faces already counted, and so placed that the common edges, if more than one is common, form one broken line, the counting, after the first face, is simply counting the sects and vertices, exclusive of the ends, of broken lines. This can be readily seen by experiment with a box, the walls of a room, or any physical solid that is a polyhedron.

In a broken line, if the ends are not counted, there is one more sect than vertex. Therefore there are as many more edges than vertices in the polyhedron as there are

broken lines to count. But there is one broken line for every face except the first face, — where the line is closed and there are the same number of edges and vertices, — and the last face, — where there are no edges or vertices to count as they have all been counted in preceding faces.

Therefore, $e = v + (f - 2)$, or $e + 2 = f + v$.

324. Radian Measure. There is another system of angular measurement that is sometimes more convenient than measurement by degrees. In this measure, the unit is a *radian*, or angle whose subtended arc, on a circle described with its vertex as center, is equal in length to the radius of the circle. Show that 2π radians are equal to a perigon, and π radians to a straight angle.

Since the circumference of a circle has a length equal to 2π times a radius (or 6.2832 r), the arc of a radian is the circumference divided by 6.2832, and the number of degrees in a radian is $\frac{360}{6.2832}$, or 57.2958° . In practical

use, this number is seldom necessary, for the fact that a straight angle equals π radians, and also equals 180° , gives a more convenient comparison between the two systems.

The use of the radian in solid geometry will be shown by two examples.

1. Using radian measure, find the area of a spherical triangle of angles 90° , 110° , and 120° , on a sphere of radius 10 in.

In radian measure, the angles, — compared with a straight angle which equals π radians, — are $\frac{1}{2}\pi$, $\frac{11}{18}\pi$, $\frac{2}{3}\pi$, and their sum is $\frac{16}{9}\pi$. But the sum of the angles of a plane triangle is π radians, so the spherical excess of this triangle is $\frac{7}{9}\pi$ radians. But the area, A , is to the area

of the sphere, 400π sq. in., as the spherical excess is to 720° , or 4π radians. Therefore

$$A = \frac{\frac{7}{2} \pi}{4 \pi} (400 \pi) = \frac{700}{9} \pi \text{ sq. in.}$$

2. If, on a sphere of radius 10 in., the sides of a spherical triangle are 12 in., 10 in., and 15 in., find the area of the polar triangle (to two decimal places).

The sides of this triangle are 1.2, 1, and 1.5 radii respectively; therefore the angles of the polar triangle are $\pi - 1.2$, $\pi - 1$, $\pi - 1.5$, radians, and its spherical excess is the sum of these angles minus π radians, or $(2\pi - 3.7)$ radians. The area is therefore

$$\frac{2\pi - 3.7}{4\pi} (400 \pi) = 200\pi - 370 \text{ or } 258.32 \text{ sq. in.}$$

In the first example, it is probably easier to use degree measure, while in the second, there is little doubt that radian measure presents less difficulty. In most cases where the lengths of arcs are expressed in length units, radian measure will be found simpler.

602. Find the area of a lune of angle $\frac{\pi}{2}$ radians, on a sphere of radius 5.

603. Find the area of $P\left(\frac{\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{8}, \frac{2\pi}{3}\right)$ on a sphere of radius 1.

604. On a sphere of radius 12 in., a spherical triangle of sides 6 in., 8 in., and 10 in. subtends a trihedral angle at the center. Find the face angles in radians, and in degrees.

605. In the triangle in the last exercise find the angles of its polar triangle, using radian measure.

606. How many radians must the angle of a lune equal if its area is half that of the sphere? one third that of the sphere?

325. Symmetry. 1. Two points are *symmetric with regard to a center* when they are on a line through the center, and are equidistant from it.

A figure is *symmetric with regard to a center* when for each point of the figure there is a corresponding point symmetric to it.

2. Two points are *symmetric with regard to a line* (or axis) when they are on a perpendicular to that line and are equidistant from it.

A figure is *symmetric with regard to a line* (or axis) when for each point of the figure there is a corresponding point symmetric to it.

3. Two points are *symmetric with regard to a plane* when they are on a perpendicular to the plane, and are equidistant from the plane.

A figure is *symmetric with regard to a plane* when for each point of the figure there is a corresponding point symmetric to it.

4. Two figures are *symmetric with regard to a plane, line, or center*, if for each point of one there is a symmetric point of the other.

UNIVERSITY OF CALIFORNIA LIBRARY
BERKELEY

Return to desk from which borrowed.
This book is DUE on the last date stamped below.

28Apr'49JA

12Aug'49MW
15Mar'50JG

18Mar'52BS
5Mar'52L

6 Feb'53HL

JAN 24 1953 LC

9Mar'54MC
FEB 23 1954 LU

24Feb'56RE

FEB 10 1956 LU

24Mar'59JS

REC'D LD

MAY 29 1959

10Nov'61EE

REC'D LD

NOV 3 1961

DEAD

27Feb'63DG

REC'D LD

FEB 19 1963

22Oct'63XS

REC'D LD

JAN 25 1964-1 PM

REC'D LD

JAN 29 '64-1 PM

YB 17322

319

304096

QA 4.57

Smith

S6

UNIVERSITY OF CALIFORNIA LIBRARY